General Theory of Sticky Prices and Optimal Monetary Policy with Path Integrals

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Abstract

This paper introduces path integrals as a mathematical tool to analytically explore distributional dynamics of sticky prices in the presence of time-varying inflation with an implied state- and time-dependent generalized hazard function. So far, no existing analytical approach can effectively be taken to analyze such an issue with time-varying growth (for instance, time-varying drift or time-varying inflation in the case of sticky prices). We explore analytical propagation of a monetary shock in the presence of time-varying inflation in a sticky-price economy in its relation to the propagation of a monetary shock in the case of zero inflation of the sticky-price economy. We also study average speed of convergence of the transition dynamics of the sticky-price economy using path integral formulation as well as the propagation of a monetary shock through the path integral transition density in its relation to the spectral (eigenvalue-eigenfunction) transition density determined by a theoretical generalization. Based on all previous theoretical results, we are finally able to study the optimal monetary policy in terms of the optimal timing for achieving the long-term inflation target set by the central banks, which we show only exists at discrete infinitely many times $T^* = \frac{n\pi}{\sigma\sqrt{2\kappa}}$ with positive integer n, where σ and κ represent the cost volatility and the curvature of the corresponding generalized hazard function, respectively, after the monetary shock.

JEL Classification Numbers: E3, E5

Key Words: sticky prices, path integral formulation, eigenvalue-eigenfunction decomposition, time-varying inflation, monetary shock, transition density, firm's reinjection, propagation of a monetary shock, optimal monetary policy.

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1 Introduction

At the beginning of the paper, we must state that the words chosen in the title of the paper "General Theory" aims to suggest that this paper is to study the dynamics of sticky prices in an economy with time-varying inflation. In a companion paper titled "Special Theory of Sticky Prices with Path Integrals", we study the dynamics of sticky prices in an economy with zero inflation. We adapt this "special-general" theory tradition from the famous "special-general" theory of relativity proposed by Albert Einstein in which the special theory of relativity applies when the gravity is not considered, while the general theory of relativity applies when the gravity cannot be ignored. In other words, we analogously compare the (time-varying) inflation in our sticky-price framework to the gravity in relativity theory.

The existing literature falls short of rigorous academic work on the dynamics (i.e., the more general dynamical theory as opposed to stationary theory) of sticky prices with generalized hazard functions and thus desperately needs a groundbreaking work, for instance, on the distributional dynamics of sticky prices with a more general state- and time-dependent generalized hazard functions in the presence of time-varying inflation, so that such an unfortunate gap between the stationary theory of sticky prices and a more general dynamical theory of sticky prices with generalized hazard functions can be finally bridged. The most recent work from Alvarez and Lippi (2022) and Alvarez, Lippi, and Oskolkov (2022) neglects studying the distributional dynamics of sticky prices in the presence of time-varying inflation with an implied state- and time-dependent generalized hazard function. Instead, their studies mainly focus on either zero inflation case or the small constant inflation which can be approximated pretty well through the zero inflation case in terms of output impulse response following a monetary shock, which implies that the generalized hazard functions used in Alvarez and Lippi (2022) and Alvarez, Lippi, and Oskolkov (2022) are all state-dependent only.

The two main distinctions between macroeconomics of sticky prices in the presence of time-varying inflation and macroeconomics of sticky prices in the case of zero inflation as studied in Alvarez and Lippi (2022) and Alvarez, Lippi, and Oskolkov (2022) are listed as follows. First, in the case of sticky prices in the presence of time-varying inflation with generalized hazard function, the implied generalized hazard function (i.e., rate of price adjustment) is both state- and time-dependent, while in the case of sticky prices with zero inflation, the implied generalized hazard function (i.e., rate of price adjustment) is just state-dependent. Second, in the former case of sticky prices in the presence of time-varying inflation with an implied state- and time-dependent generalized hazard function, firm's reinjection into the optimal return points (i.e., the points that maximize the firm's profits) right after the price adjustment has to be taken into account. While in the latter case of sticky prices with zero inflation with an implied state-dependent generalized hazard function, firm's reinjection right after the price adjustment can be ignored, because in this case the output impulse response with firm's reinjection and the output impulse response without firm's reinjection are equivalent to each other.

Figuring out how to incorporate firm's reinjection after price adjustment in the context of sticky prices with time-varying inflation and an implied state- and time-dependent generalized hazard function turns out to be a challenging task and no existing work in literature has been able to address that appropriately. The eigenvalue-eigenfunction decomposition approach used in Alvarez and Lippi (2022) and Alvarez, Lippi, and Oskolkov (2022) turns out to be unfit to deal with distributional dynamics of sticky prices in the presence of time-varying inflation with an implied state- and time-dependent generalized hazard function, because it is mathematically impossible, by the ordinary KFE formulation, to separate eigenvalues from eigenfunctions for both Kolmogorov Forward Equation (KFE) and Kolmogorov Backward Equation (KBE) which are both partial differential equations commonly used to analytically characterize transition dynamics of any state variable in macroeconomics. On the other hand, the Laplace transform approach used in Gabaix, Lasry, Lions, and Moll (2016) to study the dynamics of (top) inequality is unable to analytically address the state-dependent generalized hazard function which is a part of the partial differential equation, although Laplace transform can deal with the time-varying drift analytically in the partial differential equation. Overall speaking, we can see that eigenvalue-eigenfunction decomposition approach is unable to analytically address time-varying drift but can deal with the state-dependent generalized hazard function. By contrast, Laplace transform approach is unable to analytically address state-dependent generalized hazard function but can deal with the time-varying drift. Hence, neither eigenvalue-eigenfunction decomposition nor Laplace transform as a technical approach can be directly used to fully explore a case of macroeconomic model with both time-varying drift and an implied state- and time-dependent generalized hazard function.

Regarding our approach used for the analysis of the paper, our method of path integral formulation has its wide range of applications in macroeconomics far beyond the sticky-price models. For instance, path integral formulation also can help solve the aggregate dynamics in lumpy economies with time-dependent growth of aggregate productivity of the economy, which can be seen as a generalization of Baley and Blanco (2021). A time-dependent growth of aggregate productivity of the economy can be viewed and modeled as an important driving force for any economy that experiences a transition following an aggregate shock from its initial steady state to a new steady state which may or may not be the same as the original steady state of the economy. It is my hope that both our framework of the dynamics of sticky prices with generalized hazard functions by path integral formulation and the technical approach itself (i.e., path integrals) used to conduct the analysis will shed some light on the future works in the related areas. Indeed, the most exciting aspect of the framework of this paper is that it can be easily extended to study the transition dynamics of a wide range of macroeconomic models associated with optimal stopping property characterized by generalized hazard functions with or without time-varying growth (i.e., whether it is time-independent or time-varying productivity in lumpy investment, time-independent or time-varying inflation in sticky price, and time-independent or time-varying average return in illiquid assets and so forth). From this perspective, the macroeconomic framework and the mathematical technique itself that this paper helps to lay out encompass macroeconomics with regard to their powerful capability to analyze the various macroeconomic topics outlined above.

We make several contributions into the existing literature. First, by path integral formulation, we figure out the analytical path integral transition density of price gap and hence the analytical marginal impulse response of output following a monetary shock in the presence of time-varying inflation of a sticky-price economy with an implied state- and time-dependent generalized hazard function especially in the case of considering firm's reinjections. Second, we explore analytical propagation of a monetary shock in the presence of time-varying inflation in a sticky-price economy in its relation to the propagation of a monetary shock in the case of zero inflation of the sticky-price economy by path integral formulation. Third, we study average speed of convergence of the transition dynamics of the sticky-price economy using path integral formulation as well as the propagation of a monetary shock through the path integral transition density in its relation to the (spectral) eigenvalue-eigenfunction transition density by a theoretical generalization of path integral transition density in its relation to its corresponding eigenvalue-eigenfunction decomposition form in the presence of time-varying inflation. Finally, given all previous three contributions, we are able to study, in the context of sticky-price economy with time-varying inflation, the optimal monetary policy in terms of its optimal timing for achieving its long-term targeted inflation goal. We find that there only exists infinitely many such optimal times $T^* = \frac{n\pi}{\sigma\sqrt{2\kappa}}$ that are discrete after the monetary shock, where $n = 1, 2, 3, ...$, σ is the cost volatility and κ is the curvature of the generalized hazard function, such that if the inflation target can be achieved at those discrete times $T^* = \frac{n\pi}{\sigma\sqrt{2\kappa}}$ after the monetary shock then the monetary policy is said to be optimal.

Setup. The uncontrolled stochastic price (gap) process, after the monetary shock, for the firm is given by

$$dx(t) = \mu(t)dt + \sigma dW(t). \tag{1}$$

By uncontrolled price process, we mean this is the price (gap) process in the absence of price adjustment(s) which only happen(s) at the stopping time(s), τ(s). Here, $x(t)$ is our idiosyncratic state which is called price gap measuring the difference of log-transformed price currently charged by the firm and the optimal price that maximizes firm's profit. In this paper, we assume $x(t) \in (-\infty, \infty)$. Note that the optimal price is proportional to the cost of the firm, and therefore, the size of the fluctuation of the uncontrolled process is actually only dependent on the cost of the firm that comes from two components: inflation $-\mu(t)$ and the size of the volatility of the cost σ.

The setup for firm's problem with zero inflation is a quite standard economic environment which has been extensively studied by Nakamura and Steinsson (2010), Woodford (2009), Costain and Nakov (2011), Caballero and Engel (1999), Alvarez and Lippi (2022), Alvarez, Lippi, and Oskolkov (2022), Caplin and Spulber (1987), Caplin and Leahy (1991), and Bils and Klenow (2004). When it comes to price adjustment at stopping time τ, the firm first exits the distribution at the rate of price adjustment given by the generalized hazard function and then re-enters the distribution or is said to be reinjected into the distribution at the optimal return point x^* that maximizes firm's profit and, as a result, the price gap x becomes zero for those firms that have just reset their prices. We will show that, in the presence of time-varying inflation and sticky prices with an implied state- and time-dependent generalized hazard function, the transition dynamics of the price gap following a monetary shock with reinjections of the firms is not equivalent to the transition dynamics of the price gap without considering reinjections of the firms. Therefore, we must consider the reinjection of the firm in the presence of time-varying inflation. That is, in the presence of time-varying inflation, we must keep track of all firms even after the stopping time τ at which they reset their prices.

This paper uses generalized hazard functions $\Lambda(x, t)$ to characterize sticky-price features. Generalized hazard functions were originally developed by Caballero and Engel (1993a) and Caballero and Engel (1993b), Dotsey, King, and Wolman (1999) and further studied by Caballero and Engel (1999), Woodford (2009) and Costain and Nakov (2011). Generalized hazard functions have also been recently studied by Alvarez and Lippi (2022) and Alvarez, Lippi, and Oskolkov (2022). In general, generalized hazard function $\Lambda(x, t)$ is a function: $(x; t) \to \mathbb{R}^+ \cup \{0\}$, that maps the idiosyncratic state, i.e., the price gap x, to the rate of the price adjustment over time. Clearly, it requires $\Lambda(x^*(t), t) = 0$ because the optimal price point $x^*(t)$ closes up the price gap, i.e., price gap $x = 0$ at x^*. Since zero price gap contributes zero incentive for the firm to change its price, it follows that the rate of price adjustment $\Lambda(x, t)$ is zero at $x = x^*$. In particular, when $\mu(t) = 0$ as in Alvarez and Lippi (2022) and Alvarez, Lippi, and Oskolkov (2022), the generalized hazard function is only a function of x, i.e., $\Lambda(x, t) = \Lambda(x)$ with zero inflation and optimal return point in the case of zero inflation is $x^* = 0$. However, when inflation is time-varying, the generalized hazard function $\Lambda(x, t)$ is a function of both x and t and the optimal return point $x^*(t)$ in the presence of time-varying inflation is no longer zero and should be time-dependent as well.

2 The KFE-based differential operator approach and path integral transition density

The KFE formulation with time-varying inflation and the corresponding eigenvalue-eigenfunction decomposition can be better understood by a differential operator $\mathcal{A}$. We will technically discuss in this subsection why the traditional KFE-based differential operator approach would fail in analytically dealing with such a case with time-varying inflation $\mu(t)$. To do that, we first define a differential operator associated with the corresponding KFE (i.e., the transition matrix corresponding to the KFE) , $\mathcal{A}_t$, as

$$\mathcal{A}_t u = \mu(t)u_x + \frac{\sigma^2}{2}u_{xx} - \Lambda(x,t)u \tag{2}$$

and whose adjoint is given by

$$\mathcal{A}_t^* p = -\mu(t)p_x + \frac{\sigma^2}{2}p_{xx} - \Lambda(x,t)p. \tag{3}$$

Obviously, $\mathcal{A}_t \neq \mathcal{A}_t^*$ and hence the differential operators in the case of general theory of sticky prices with time-varying inflation, $\mathcal{A}_t$ and $\mathcal{A}_t^*$, are not self-adjoint and therefore the eigenvalues of both transition matrices $\mathcal{A}_t$ and $\mathcal{A}_t^*$ can be complex. So, in the general theory of sticky prices with time-varying inflation, we must transform operator $\mathcal{A}_t$ which is not self-adjoint into a self-adjoint operator $\mathcal{B}_t$ before we can proceed to conduct any meaningful analysis. In fact, the differential operators $\mathcal{A}_t$ and $\mathcal{B}_t$ are closely related and can be easily transformed between one and another. For instance, if we let

$$\mathcal{B}_t v = \frac{\sigma^2}{2}v_{xx} - \left[\Lambda(x,t) + \frac{1}{2}\frac{\mu^2(t)}{\sigma^2}\right]v, \tag{4}$$

such that the $p = ve^{-\frac{\mu(t)}{\sigma^2}x}$, then the differential operator $\mathcal{A}_t$ which is not self-adjoint can be completely transformed into the self-adjoint differential operator (i.e., self-adjoint transition matrix) $\mathcal{B}_t$. That is, the KFE without the source term

$$p_t = \mathcal{A}_t^* p \tag{5}$$

is equivalent to the KFE

$$v_t = \mathcal{B}_t v, \tag{6}$$

where $p = ve^{-\frac{\mu(t)}{\sigma^2}x}$.

Note that here comes a problem regarding the eigenvalues λ and the corresponding eigenfunctions (i.e., the eigenvectors) ϕ of transition matrix $\mathcal{B}_t$ with the standard spectral decomposition approach. We know that (by definition of the eigenvalues and eigenvectors of a matrix in matrix algebra) the eigenvalues and eigenfunctions are typically determined by

$$\mathcal{B}_t \phi = \lambda \phi. \tag{7}$$

However, here, due to the fact that we have the time-dependent differential operator (i.e., time-dependent transition matrix) $\mathcal{B}_t$ instead of the time-independent differential operator or transition matrix $\mathcal{B}$ as suggested by equation (11), the eigenvalues and the corresponding eigenfunctions of the transition matrix $\mathcal{B}_t$ are no longer separable. Therefore, in the presence of time-varying inflation $\mu(t)$, the equation (14) which determines the eigenvalues and the corresponding eigenfunctions of transition matrix $\mathcal{B}_t$ cannot be derived from its previous equation (13). This explains the reason why the KFE-based traditional differential operator approach fails in analytically delivering any theoretical eigenvalue-eigenfunction decomposition result in the presence of time-varying inflation. We see the key reason is because the transition matrix $\mathcal{B}_t$ is time-dependent in a sticky-price economy characterized by a generalized hazard function with time-varying inflation. As a result, we need to turn our focus to somewhere else to seek an useful analytical tool to help us to overcome this difficulty. It turns out that the path integral transition density of sticky price which is not in the form of eigenvalue-eigenfunction decomposition is our desired solution that can help us overcome this difficulty. This paper is mainly about this path integral transition density in the presence of time-varying inflation and how to use it in our further analysis.

More importantly, let us take a moment to think about what the time-dependent transition matrix $\mathcal{B}_t$ means to us and what that means to the resulting transition density. For simplicity, let us take the two-state Markov chain in discrete time as an example to investigate how the time-dependent transition matrix affects its corresponding transition density, because having a good understanding of it is key to this paper. In a standard first-year macro class (and very much likely to be among the very first several classes of the first semester), we have been taught that the the standard transition matrix in the case of two-state Markov chain in discrete time Π is independent of time, meaning the standard model assumes that the transition matrix in the case of two-state Markov chain is time-independent. And that is why there is typically no need to put a subscript t with the transition matrix Π in a standard model. However, once we deviate a little bit from the standard two-state Markov chain, we find that in reality it is very likely that the transition matrix Π_t is time-dependent, just like the time-dependent transition matrix $\mathcal{B}_t$ in our model discussed above.

Now, we want to ask, in the case of the two-state Markov chain in discrete time, if the (2×2) transition matrix Π_t is time-dependent, then what does that mean to the resulting transition density of the two-state Markov process relative to the case of time-independent two-state transition matrix Π ? If we understand the difference here in the two-state Markov process, we will understand our model regarding time-dependent transition matrix $\mathcal{B}_t$ versus time-independent transition matrix $\mathcal{B}$ as well, because they are actually the same thing. Given the initial stationary distribution of the two-state Markov process, π_0, we can express the time evolution of the density of the two states as $\pi_t = \Pi^t \pi_0$ in the case of time-independent two-state transition matrix Π. But when the two-state transition matrix becomes time-dependent, i.e., from time 0 to time τ the transition matrix is Π, and from time τ to time t the corresponding transition matrix is Ω, where $\Pi \neq \Omega$ and here all times are discrete non-negative integers, the resulting time evolution of the density in this latter case is written as $\pi_t = 1_{\{t \leq \tau\}} \Pi^\tau \pi_0 + 1_{\{t > \tau\}} \Omega^{t-\tau} \Pi^\tau \pi_0$.

Now, in terms of the transition density in this two-state Markov process, we know that the transition density is given by the row elements of the two-state (2×2) transition matrix of Π from time 0 to τ and transition matrix of Ω from time τ to t. For example, the vector consisting of first-row elements of transition matrix Π is the transition density of this two-state Markov process which is conditional on the first state in time $t \in [0, \tau]$, $\Pi(s_{t+1} \in \{1, 2\} | s_t = 1)$. Similarly, the vector consisting of first-row elements of transition matrix Ω is the transition density of the two-state Markov process which is conditional on the first state in time $t \in (\tau, \infty)$, $\Omega(s_{t+1} \in \{1, 2\} | s_t = 1)$. We therefore find out that depending on the time, there exists two different transition density functions which are both conditional on the first state, as discussed above, which explains why there does not exist eigenvalue-eigenfunction decomposition in such a time-dependent transition matrix case, because for eigenvalue-eigenfunction decomposition to be possible the transition density function has to be unique for all time t. The same logic applies to our case of time-dependent transition matrix $\mathcal{B}_t$, since there does not exist a unique transition density function that works for all t with time-dependent transition matrix $\mathcal{B}_t$, it follows that there does not exist eigenvalue-eigenfunction decomposition of the time-dependent transition matrix $\mathcal{B}_t$.

Path integral transition density. Based on the discussions above, it naturally comes down to the question of how could the path integral transition density help ? The path integral transition density can help in two ways. First, we will show that the path integral transition density of sticky price does not take eigenvalue-eigenfunction form and hence the path integral transition density of price gap can be obtained even with time-dependent transition matrix $\mathcal{B}_t$. Second, we will also show that if we assume that the time-dependent transition matrix $\mathcal{B}_t$ only exists between time 0 and time T with time-independent transition matrix $\mathcal{B}$ at the two endpoints $t = 0$ and $t = T$, then it will turn out that the path integral transition density of price gap going from x at time 0 to y at time T implies the (generalized) eigenvalue-eigenfunction form. This assumption works especially well for the monetary policy. For example, consider a case where the time-varying inflation $\mu(t)$ only exists between time 0 and time T with zero inflation at the two endpoints $t = 0$ and $t = T$. That is, $\mu(0) = \mu(T) = 0$ implies the time-independent transition matrix $\mathcal{B}$ at the two endpoints $t = 0$ and $t = T$. Here, we can interpret time $t = 0$ as the initial steady state with zero inflation and time $t = T$ as the time at which the central banks achieve the long-term inflation target goal. For simplicity, we assume central banks' long-term inflation target is zero inflation. Then, studying the propagation of a monetary shock at time $t = T$ is important for the central banks. Given the assumptions, we have

$$K^{\mathcal{B}_t}(y|x) = \sum_{j=1}^{\infty}\sum_{i=1}^{\infty} \lambda_{ji}(T)\phi_j(y)\phi_i(x), \tag{8}$$

where on the left hand side $K^{\mathcal{B}_t}(y|x)$ is the path integral transition density of price gap going from x at time 0 to y at time T and on the right hand side $\sum_{j=1}^{\infty}\sum_{i=1}^{\infty} \lambda_{ji}(T)\phi_j(y)\phi_i(x)$ is its implied corresponding transition density of price gap of the (generalized) eigenvalue-eigenfunction decomposition form. Note that, unlike in the case of time-independent transition matrix $\mathcal{B}$ in which there does exist the standard transition density of the eigenvalue-eigenfunction decomposition form implied by the path integral transition density, in the case of time-dependent transition matrix $\mathcal{B}_t$ there does not exist the standard transition density of the eigenvalue-eigenfunction form implied by the path integral transition density, but rather we call it (generalized) eigenvalue-eigenfunction decomposition form, which is different from the standard eigenvalue-eigenfunction decomposition form in the case of time-independent transition matrix $\mathcal{B}$.

The main contribution of the paper. We logically discuss the contribution of this paper here. The main contribution of the paper starts with an observation that the path integral transition density of sticky-price gap in the presence of time-varying inflation, $K^{\mathcal{B}_t}(y|x) = \sum_{j=1}^{\infty}\sum_{i=1}^{\infty}\lambda_{ji}(T)\phi_j(y)\phi_i(x)$, is asymmetric, while its corresponding path integral transition density of sticky-price gap in the absence of time-varying inflation (i.e., with zero inflation), (see the companion paper,i.e., the special theory), $K^{\mathcal{B}}(y|x) = \sum_{j=1}^{\infty} e^{-\lambda_j t}\phi_j(y)\phi_j(x)$, is symmetric. This can be easily seen by noticing that $K^{\mathcal{B}_t}(y|x) \neq K^{\mathcal{B}_t}(x|y)$ but $K^{\mathcal{B}}(y|x) = K^{\mathcal{B}}(x|y)$ from the eigenvalue-eigenfunction form implied by the path integral transition density above. That is, $\sum_{j=1}^{\infty}\sum_{i=1}^{\infty}\lambda_{ji}(T)\phi_j(y)\phi_i(x) \neq \sum_{j=1}^{\infty}\sum_{i=1}^{\infty}\lambda_{ji}(T)\phi_j(x)\phi_i(y)$ as long as $j \neq i$, but $\sum_{j=1}^{\infty} e^{-\lambda_j t}\phi_j(y)\phi_j(x) = \sum_{j=1}^{\infty} e^{-\lambda_j t}\phi_j(x)\phi_j(y)$.

Hence, based on this observation, we further learn that one of the most implications of this asymmetric transition density $K^{\mathcal{B}_t}(y|x)$ with time-varying inflation vs. symmetric transition density $K^{\mathcal{B}}(y|x)$ with zero inflation is that the asymmetric transition density of sticky-price gap with time-varying inflation causes the average speed of the convergence of the marginal output impulse response in the presence of time-varying inflation to be faster than that of the marginal output impulse response in the absence of time-varying inflation (i.e., with zero inflation). Or equivalently, the asymmetric transition density of sticky-price gap with time-varying inflation causes a larger amplification effect for the marginal output impulse response in the presence of time-varying inflation than that for the marginal output impulse response in the absence of time-varying inflation (i.e., with zero inflation), which is not good for the sticky-price economy with time-varying inflation in transition. Thus, the goal will be to make the transition density of sticky-price gap in the presence of time-varying inflation as symmetric as possible, so that the amplification effect could be dampened.

Next, we theoretically argue for an optimal monetary policy that could be used by central banks to correct this asymmetric property of the transition density of sticky-price gap in the presence of time-varying inflation. Here comes a key point, which is that if we can take a closer second look at the asymmetric transition density $K^{\mathcal{B}_t}(y|x)$ in its eigenvalue-eigenfunction form and the symmetric transition density $K^{\mathcal{B}}(y|x)$ in its eigenvalue-eigenfunction form above, we can easily find that the asymmetry of $K^{\mathcal{B}_t}(y|x)$ only occurs when the $j \neq i$ with the time-varying inflation. If

$j = i$, then the asymmetry will be transformed into symmetry for the $K^{\mathcal{B}_t}(y|x)$ with time-varying inflation. That is, if $j = i$, then the transition density of sticky-price gap with time-varying inflation $K^{\mathcal{B}_t}(y|x)$ becomes symmetric and takes the form of linear transformation of the symmetric transition density of sticky-price gap with zero inflation $K^{\mathcal{B}}(y|x)$, i.e., if $j = i$, then $K^{\mathcal{B}_t}(y|x) = \sum_{j=1}^{\infty} \lambda_{jj}(T)\phi_j(y)\phi_j(x) = \mathcal{L}\big(K^{\mathcal{B}}(y|x)\big) = \mathcal{L}\Big(\sum_{j=1}^{\infty} e^{-\lambda_j t}\phi_j(y)\phi_j(x)\Big)$ for all $t = T$, where $\mathcal{L}(\cdot)$ means linear transformation.

Finally, we give suggestions to the central banks in terms of how to practically achieve $j = i$ so that the amplification effect due to the asymmetry can be dampened. To achieve this goal, the central banks must use their monetary policy as the monetary tool to set discrete times $T^* = \frac{n\pi}{\sigma\sqrt{2\kappa}}$ at which the long-term zero inflation target is achieved after the monetary shock. Note here for simplicity we assume the long-term inflation target for central banks is zero inflation. And here σ is the volatility of the cost, κ is the curvature of the generalized hazard function, and $n = 1, 2, 3, ...$

The organization of the paper. In order to achieve those goals mentioned as the contribution of the paper, the paper is organized as follows. We discuss the background knowledge that is necessary for us to better understand this paper in section 1 and section 3 in terms of the setup, literature, the monetary shock in our model and the impulse response function with section 2 as an exception. We view section 2 is a very important section for this paper in a way that helps relate the exiting literature to the content of this paper. We also outline an analytical method for how to incorporate the firm's reinjection into the path integral transition density of sticky-price gap with time-varying inflation in section 2.

We formally introduce path integral formulation in the context of the sticky prices with generalized hazard functions in section 4 and finish this section with an example to show what the resulting path integral transition density of sticky-price gap in the case of zero inflation looks like in terms of its symmetry and the functional form. In section 5, we study the path integral transition density of sticky-price gap in the presence of time-varying inflation in both cases with or without firm's reinjection. We show the path integral transition density of sticky-price gap in the presence of time-varying inflation with firm's reinjection is totally distinct from that without firm's reinjection in section 5.

Starting section 6, our goal is on the optimal monetary policy analysis. We establish a relation between the marginal impulse response in the case of zero inflation and the marginal output impulse response in the presence of time-varying inflation with firm's reinjection in section 6.1. We study under what circumstance the path integral transition density of sticky-price gap in the presence of time-varying inflation becomes symmetric and takes linear transformation of the path integral transition density of sticky-price gap with zero inflation in section 6.2. In section 7, we generalize the path integral transition density of sticky-price gap in its relation to the transition density of sticky-price gap of the form of eigenvalue-eigenfunction decomposition in the presence of time-varying inflation.

Based on this generalization, we finally are able to re-express the path integral transition density of sticky-price gap in the presence of time-varying inflation in terms of the generalized eigenvalue-eigenfunction form and hence also be able to re-express the marginal output impulse response using the generalized eigenvalue-eigenfunction form. Importantly, with the help of this generalization, we can clearly see how and to what extend the optimal monetary policy studied earlier in the paper relates to the eigenvalue-eigenfunction form. We end our analysis with a set of average speeds of convergence for the marginal output impulse responses in three scenarios, namely, the zero inflation scenario, the time-varying inflation scenario with $T = T^*$, the time-varying inflation scenario with $T \neq T^*$ and compare them.

Firms' reinjections and path integral transition density. We next outline the analytical steps used to tackle the main challenge of the dynamics of time-varying inflation and sticky prices with an implied state- and time-dependent generalized hazard function, which are heavily dependent on the transition density version of the KFEs. First, we intuitively illustrate why eigenvalue-eigenfunction decomposition method used in Alvarez and Lippi (2022) could not be applied to the time-varying inflation case as follows. When the inflation is time-varying, i.e., $\mu(t)$ is not zero anymore but a function of real time t, the KFE formulation of the problem is written as

$$\partial_t p(x,t) = -\mu(t)\partial_x p(x,t) + (\sigma^2/2)\partial_x^2 p(x,t) - \Lambda(x,t)p(x,t) \tag{9}$$

if not considering firm's reinjection. Moreover, if considering firm's reinjection, the corresponding version of KFE is given by

$$\partial_t p(x,t) = -\mu(t)\partial_x p(x,t) + (\sigma^2/2)\partial_x^2 p(x,t) - \Lambda(x,t)p(x,t) + \Lambda(x,t)\delta(x(t) - x^*(t)), \tag{10}$$

where the last term $\Lambda(x,t)\delta(x(t)-x^*(t))$ captures the reinjections of the firms into the optimal return point $x^*(t)$ at the rate of $\Lambda(x,t)$ upon price adjustment and therefore $\delta(x(t)-x^*(t))$ is the Dirac Delta function at $x^*(t)$. For both cases, eigenvalue-eigenfunction decomposition cannot be applied to the KFE(s) above simply because the time-varying drift or inflation $\mu(t)$ makes it impossible to write the solution to the KFE with time-varying drift in terms of eigenvalue-eigenfunction decomposition, (i.e., $p(x,t)=\sum_{j=1}^{\infty} e^{\lambda_j t}\phi_j(x)$, where λ_j and $\phi_j(x)$ are eigenvalues and eigenfunctions, respectively, is impossible for the solution to the KFE with time-varying drift). Now, since the path integral transition density is a particular type of time evolution of the density given any arbitrary initial condition at s that does not necessarily have to be equal to the the initial condition at $t=0$ in general, it follows that the path integral transition density of price gap in the presence of time-varying inflation $\mu(t)$ without considering firm's reinjection, $K^{\mu(t)}(y|x)$, also satisfies the KFE above; that is, without considering firm's reinjection, it is given by

$$\partial_t K^{\mu(t)}(y|x) = -\mu(t)\partial_y K^{\mu(t)}(y|x) + (\sigma^2/2)\partial_y^2 K^{\mu(t)}(y|x) - \Lambda(y,t)K^{\mu(t)}(y|x), \quad (11)$$

and with firm's reinjection, the path integral transition density based KFE is correspondingly given by

$$\begin{aligned}\partial_t \mathcal{K}^{\mu(t)}(y|x) = &-\mu(t)\partial_y \mathcal{K}^{\mu(t)}(y|x) + (\sigma^2/2)\partial_y^2 \mathcal{K}^{\mu(t)}(y|x) - \Lambda(y,t)\mathcal{K}^{\mu(t)}(y|x) \\ &+ \Lambda(y,t)\delta(y(t)-y^*(t)),\end{aligned} \quad (12)$$

where $\mathcal{K}^{\mu(t)}(y|x)$ denotes the path integral transition density of price gap in the presence of time-varying inflation with an implied state- and time-dependent generalized hazard function and with firm's reinjection. Given the background discussion above, the analytical steps used to conduct our analysis for transition density of price gap in the case of sticky prices and time-varying inflation with firm's reinjection is as follows. In the first step, we show path integral formulation allows us to analytically obtain the path integral transition density of price gap in the presence of time-varying inflation, $K^{\mu(t)}(y|x)$, a solution to the KFE (11) without considering firm's reinjection. In the second step, we show that the path integral transition density of price gap in the presence of time-varying inflation without firm's reinjection $K^{\mu(t)}(y|x)$ takes the multiplicative form of path integral transition density of price gap in the case of zero inflation, $K^0(y|x)$, and some other terms that aim to account for the impact of the time-varying inflation $\mu(t)$, where $K^0(y|x)$ solves

$$\partial_t K^0(y|x) = (\sigma^2/2)\partial_y^2 K^0(y|x) - \Lambda(y)K^0(y|x) + \Lambda(y)\delta_0(y), \tag{13}$$

which, as we will show, shares the same solution as the solution to the case without considering firm's reinjection in the case of zero inflation, i.e.,

$$\partial_t K^0(y|x) = (\sigma^2/2)\partial_y^2 K^0(y|x) - \Lambda(y)K^0(y|x). \tag{14}$$

In the third step, to transform the path integral transition density of price gap in the presence of time-varying inflation without firm's reinjection $K^{\mu(t)}(y|x)$ into the path integral transition density of price gap in the presence of time-varying inflation with firm's reinjection, $\mathcal{K}^{\mu(t)}(y|x)$, we just need to replace the $K^0(y|x)$ component in the multiplicative form of $K^{\mu(t)}(y|x)$ with the eigenvalue-eigenfunction transition density of price gap, i.e., $\mathcal{K}^0(y|x)$, where τ is the stopping time at which the firm resets price to the optimal return point $y^*(\tau) \neq 0$ in the presence of time-varying inflation and therefore $\mathcal{K}^0(y|x)$ solves

$$\partial_t \mathcal{K}^0(y|x) = (\sigma^2/2)\partial_y^2 \mathcal{K}^0(y|x) - \Lambda(y)\mathcal{K}^0(y|x) + \Lambda(y)\delta(y^*(\tau)). \tag{15}$$

As a result, we can obtain our desired path integral transition density of price gap in the presence of time-varying inflation of a sticky-price economy with an implied state- and time-dependent generalized hazard function and with firm's reinjection, $\mathcal{K}^{\mu(t)}(y|x)$, by the three steps outlined above.

Stationary distribution. We study the analytical marginal impulse response of output following a monetary shock. With quadratic generalized hazard function at time 0 before shock and time-varying inflation arrive, $\Lambda(x) = \kappa x^2$, we first must solve for the steady state of the distribution of the price gap with that quadratic hazard function. Simply note that the time-independent KFE characterizing the stationary distribution of the price gap with a quadratic hazard function is written as $\kappa x^2 f(x) = \frac{\sigma^2}{2} f''(x)$, where $f(x)$ is the stationary distribution of price gap x and σ is a measure of cost uncertainty of the stationary economy (i.e., the volatility of the initial steady-state economy). The solution to this time-independent KFE takes the following form

$$
\begin{aligned}
f(x) &= e^{-\sqrt{\kappa/2\sigma^2}x^2}\sum_{j=0}^{\infty} a_{2j}x^{2j} \\
&= a_0 e^{-\sqrt{\kappa/2\sigma^2}x^2} + e^{-\sqrt{\kappa/2\sigma^2}x^2}\sum_{j=1}^{\infty} a_{2j}x^{2j}
\end{aligned}
\tag{16}
$$

where a_0 is a normalization factor and is determined by the normalization of density $f(x)$ (i.e., $\int_{-\infty}^{\infty} f(x)dx = 1$) and the coefficients a_{2j} are recursively given by

$$
a_{2j+2} = \frac{4\sqrt{\kappa/2\sigma^2}j + \sqrt{\kappa/2\sigma^2}}{(2j+1)(j+1)}a_{2j}, \tag{17}
$$

note that all coefficients a_{2j} will be completely determined by normalization factor a_0. We thus can write the output marginal impulse response function as

$$
\mathcal{Y}(t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(-y)K^0(y|x)f'(x)dxdy,
$$

where $K^0(y|x)$ is the transition density of price gap following a monetary shock in the case of zero inflation, that is

$$
K^0(y|x) = \left[\frac{\sqrt{2\kappa}}{2\pi\sigma\sinh\sqrt{2\kappa}\sigma t}\right]^{1/2} e^{-\frac{\sqrt{2\kappa}}{2\sigma}\left[\frac{\left(x^2+y^2\right)\cosh\sigma\sqrt{2\kappa}t - 2xy}{\sinh\sigma\sqrt{2\kappa}t}\right]},
$$

and $f'(x)$ is the derivative of the invariant density of price gap which is given by

$$
\begin{aligned}
f'(x) = a_0(-2)\sqrt{\kappa/2\sigma^2}xe^{-\sqrt{\kappa/2\sigma^2}x^2} + (-2)\sqrt{\kappa/2\sigma^2}xe^{-\sqrt{\kappa/2\sigma^2}x^2}\sum_{j=1}^{\infty} a_{2j}x^{2j} \\
+ e^{-\sqrt{\kappa/2\sigma^2}x^2}\sum_{j=1}^{\infty}(2j)a_{2j}x^{2j-1}.
\end{aligned}
$$

3 Monetary shock and impulse response

This paper aims to explore the distributional dynamics of a sticky-price economy with generalized hazard functions following a monetary shock. In terms of a monetary shock studied by this paper, we consider a parallel shift in all price gaps. The rationale is that, under the specific assumptions this paper follows, the parallel shift in the level of money supply maps into a parallel shift in nominal wages and thus parallel shift in all price gaps. When it comes to uncertainty shocks, however, we do not have the parallel shift in all price gaps anymore. Specifically, with uncertainty shocks, the dispersion of the initial steady-state distribution of the price gaps will be changed without a parallel shift in distribution of the price gaps. Furthermore, in a sequence of uncertainty and monetary shocks, we will have the mixture of both parallel shift in the price gaps and the changes in the dispersion of the price gaps. Throughout the paper, we mainly discuss how a monetary shock could drive the distributional dynamics of a sticky-price economy in the case of time-varying inflation.

A monetary shock. Specifically, when it comes to the distributional dynamics of the price gap following a monetary shock, we take a similar approach as in Alvarez, Lippi, and Souganidis (2023); that is, we consider a perturbation ν of the stationary density of price gap $f(x)$, or equivalently, we define the initial condition of the density of price gap right after the monetary shock of size δ, $f_0(x)$, as

$$f_0(x) = f(x) + \delta\nu(x), \tag{18}$$

where $\int_{-\infty}^{\infty} \nu(x)dx = 0$.

In the spirit of Alvarez, Lippi, and Souganidis (2023) and in the context of small monetary shock characterized by the small size of the monetary shock δ (i.e., small δ), there is a particular perturbation focused by this paper which is the one corresponding to an unanticipated aggregate nominal shock that changes the nominal costs of all firms by an amount δ, so that the initial condition for the density of price gap before any decision is taken is

$$f_0(x) = f(x + \delta), \tag{19}$$

which is a special case of $f_0(x) = f(x) + \delta\nu(x)$ where $\nu(x) = f'(x)$, which follows from the fact that $f(x+\delta) = f(x) + \delta f'(x) + \mathcal{O}(\delta)$ implied by Taylor expansion of $f(x+\delta)$. The interpretation of such an initial condition of the density of price gap is that after the monetary shock of size δ the nominal cost jumps immediately and hence the value of the price gap x for each firm jumps from x to $x - \delta$. Hence, in this paper, the signed measure $\hat{f}(x) = f(x+\delta) - f(x)$ describing the deviation of the initial condition of the density of price gap from the stationary density of price gap right after the monetary shock to the stationary density is given by $\hat{f}(x) = \delta f'(x) + \mathcal{O}(\delta)$. Note that the impulse response function is basically the expected value of any variable of interest computed on this signed measure and that is why this signed measure is an important component of the impulse response function of any variable of our concern. For the consideration of the marginal version of the monetary shock, we thus have

$$\left.\frac{\partial \hat{f}(x)}{\partial \delta}\right|_{\delta \to 0} = f'(x), \tag{20}$$

which is the version of the signed measure that will be used in the marginal output impulse response function of this paper.

Impulse response function of output. We use output impulse response function

$$Y(t;\delta) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(-y)q_t(y|x)dyd\hat{F}(x), \tag{21}$$

where $F(x)$ is the corresponding cumulative density and thus the corresponding marginal version of output impulse response function as $\delta \to 0$, $\mathcal{M}(t)$, is given by

$$\mathcal{M}(t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(-y)q_t(y|x)f'(x)dxdy, \tag{22}$$

where $q_t(y|x)$ is the transition density of price gap from x at time 0 to y at time t and $-y$ is due to the fact that output is inversely proportional to output gap.

Related literature. Almost all analytical works in existing literature are only focused on zero inflation or constant or steady-state inflation. For example, some results have been given by Alvarez and Lippi (2022) for a simplified problem featuring symmetry of the firms' decision (a symmetric $p(x,t)$ density) and lack of drift (i.e., zero inflation) in the firm's state x, assumed to follow a driftless diffusion $dx = \sigma dW$ where W is a Brownian motion. Symmetry and lack of drift turn out to be analytically convenient because they give rise to a situation that allows analytical eigenvalue-eigenfunction decomposition to be a feasible analytical approach to explore such a simplified case. Also see Taylor (1980), Calvo (1983), Golosov and Lucas (2007), Nakamura and Steinsson (2010), Caballero and Engel (2007), Midrigan (2011), Bhattarai and Schoenle (2014), Alvarez and Lippi (2014), and Alvarez and Lippi (2020).

The adjustment choice formulated as a generalized hazard function is also a popular idea in literature but the formulation of this paper nests fully both state-dependent adjustment and time-dependent adjustment. In existing literature, most works either study the zero-drift case $\mu(t) = 0$ as mentioned above with a generalized hazard function $\Lambda(x)$ which is only state-dependent, see Alvarez and Lippi (2022), and thus yields a symmetric and stationary environment, or study constant-drift case $\mu(t) = \mu \neq 0$ and thus piece-wise hazard is studied by Baley and Blanco (2021), which in this case will yield an asymmetric and stationary environment. For both previous cases in existing literature, there is a steady-state distribution of price gap characterized by a time-independent KFE. Its characterization is key for (i) studying the propagation of aggregate shocks, i.e., monetary shocks or aggregate productivity shocks, (ii) deriving sufficient statistics that characterize marginal impulse response function, and (iii) establishing mappings to the micro-data. Furthermore, the existing standard analytical approach is enough to analyze both previous cases, whether it is by eigenvalue-eigenfunction decomposition or Laplace transform method. See Caballero and Engel (2007), Alvarez, Bihan, and Lippi (2016), Baley and Blanco (2021), Alexandrov (2020), Hansen and Scheinkman (2009), and Gabaix et al. (2016). The state dependence has been studied both theoretically and empirically in literature by Barro (1972), Sheshinski and Weiss (1977), Dixit (1991), Golosov and Lucas (2007), Fougere, Bihan, and Sevestre (2007), Dias, Marques, and Silva (2007), Eichenbaum, Jaimovich, and Rebelo (2011), and Gautier and Saout (2015). There are also many papers with numerical results along this line of work though across durable consumption, saving portfolios, mortgage refinance, monetary policy with portfolio frictions and investment, see Eberly (1994), Attanasio (2000), Stokey (2009), Caballero and Engel (1999), Baley and Blanco (2021), Alvarez, Guiso, and Lippi (2012), Abel, Eberly, and Panageas (2013), Alvarez, Atkeson, and Edmond (2009), and Silva (2012). However, the literature lacks its theoretical counterpart. Hence, this paper aims to fill such a gap.

4 Introduction to path integral formulation with generalized hazard functions

This section introduces path integrals in the context of sticky prices and generalized hazard functions to explore the distributional dynamics of the sticky-price economy following a monetary shock in the case of both zero inflation and time-varying inflation. The theoretical framework of path integrals outlined in this section generally works not only for an economy of sticky prices in the case of zero inflation with an implied state-dependent-only generalized hazard function but also for a sticky-price economy in the presence of time-varying inflation with an implied state- and time-dependent generalized hazard function. The only difference between the two cases is that the generalized hazard function in the former case of zero inflation takes the form of $\Lambda(x)$ and the generalized hazard function in the latter case of time-varying inflation takes the form of $\Lambda(x, t)$.

The fundamental idea about path integrals (or path integral formulation) for studying macroeconomic dynamics of sticky prices with generalized hazard functions following a monetary shock is to analytically obtain the transition density of price gap x going from x_a at time t_a to x_b at time t_b, $K(b|a)$, or simply, the transition probability that price gap ends up being x_b at time t_b given it starts with x_a at time t_a. Now, imagine the following thought experiment. (Also see page 59 of Feynman and Hibbs (1965), i.e., the Gaussian Integrals section). First, following a shock, let us denote a deterministic time path of price gap x from x_a at time t_a to x_b at time t_b by $\bar{x}(t)$ which is the deterministic time path of x based on the principle of least action, and the actual time path of the price gap x from t_a to t_b by $x(t)$, where $t \in [t_a, t_b]$. Then, the actual time path of price gap over the transition period $x(t)$ can be written as the sum of the deterministic time path $\bar{x}(t)$ of least action and the deviation of the actual time path $x(t)$ from the deterministic path $\bar{x}(t)$ of least action, namely, $y(t)$, as

$$x(t) = \bar{x}(t) + y(t) \tag{23}$$

that is, instead of defining a point on the path by its distance $x(t)$, we measure instead the deviation $y(t)$ from the least-action deterministic path $\bar{x}(t)$. Given the transitional time path from t_a to t_b, both the actual and the least-action deterministic time path of price gap from the transition period t_a to t_b following a shock have the same initial and terminal locations because they are actually both the transitional time paths between x_a at time t_a and x_b at time t_b (i.e., fixing end points but varying the path in-between), that is,

$$x(t_a) = \bar{x}(t_a) = x_a$$

and

$$x(t_b) = \bar{x}(t_b) = x_b,$$

and therefore, based on $x(t) = \bar{x}(t) + y(t)$, we get

$$y(t_a) = y(t_b) = 0,$$

that is, the deviation of the actual time path of price gap from the least-action deterministic time path of the price gap at the initial and terminal locations x_a and x_b, respectively, are both equal to zero.

In between these end points $y(t)$ can take any form. Since the least-action deterministic path $\bar{x}(t)$ is non-random and can always be solvable according to the principle of least action in which Euler-Lagrange (EL) equation applies, any variation by a perturbation in the alternative path $x(t)$ is equivalent to the associated variation in $y(t)$. Thus, in a path integral, the path differential $\mathcal{D}x(t)$ can be replaced by $\mathcal{D}y(t)$, i.e., $\mathcal{D}x(t) = \mathcal{D}y(t)$, and the path $x(t)$ by $\bar{x}(t) + y(t)$. Here, we use $\mathcal{D}$ to denote path differential rather than the ordinary differential d used in the standard calculus.

In this form, $\bar{x}(t)$ is the least-action deterministic path for the integration which is analytically given by EL equation. Moreover, the stochastic path $y(t)$ is restricted to take the value 0 at both end points. This substitution leads to a path integral independent of end-point positions. See Page 59 of Feynman and Hibbs (1965). In what follows, we specifically illustrate how to use path integral formulation to analytically explore the transition dynamics of a sticky-price economy following a monetary shock for the cases where zero inflation $\mu(t) = 0$ implying state-dependent generalized hazard function $\Lambda(x)$ and time-varying inflation $\mu(t)$ implying state- and time-dependent generalized hazard function $\Lambda(x, t)$ associated with the volatility of the economy, σ.

It follows from the definition of the path integral formulation Feynman and Hibbs (1965) that the path integrals, given our economic settings, are formulated by the following integral for the transition density, $K(x_b, t_b; x_a, t_a) = K(b|a)$, which represents the transition density of price gap going from x_a at time t_a to x_b at time t_b as

$$K(b|a) = \int_{x_a}^{x_b} e^{-\frac{1}{\sigma^2}\int_{t_a}^{t_b} L(\dot{x}, x, \tau)d\tau} \mathcal{D}x(\tau),$$

where $L(\dot{x}, x, t) = \frac{1}{2}\dot{x}^2(t) + \sigma^2\Lambda(x, t)$ is the Lagrangian, or equivalently, it is rewritten as

$$K^0(b|a) = \int_{x_a}^{x_b} e^{-\frac{1}{\sigma^2}\int_{t_a}^{t_b}\left[\frac{1}{2}\dot{x}^2(\tau)+\sigma^2\Lambda(x)\right]d\tau}\mathcal{D}x(\tau) \tag{24}$$

with zero inflation and

$$K^{\mu(t)}(b|a) = w_{t_b}(x_b)\int_{x_a}^{x_b} e^{-\frac{1}{\sigma^2}\int_{t_a}^{t_b}\left[\frac{1}{2}\dot{x}^2(\tau)+\sigma^2\Lambda(x,\tau)\right]d\tau}\mathcal{D}x(\tau) \tag{25}$$

in the presence of time-varying inflation. Here, $x(t)$ denotes the any possible transitional path of price gap x from time t_a to time t_b. $\mathcal{D}$ explicitly refers to the fact that the integral is taken with respect to all the possible paths of x between x_a and x_b.

One of the biggest shortcomings of Brownian motion (Weiner) processes based on which KFE is formulated is that the Brownian stochastic process is not differentiable with respect to time. In Section 5 we show that any Brownian stochastic process can be equivalently formulated by path integrals by an equivalence of KFE and path integral formulation in this regard. Here, we aim to show that any path integral formulated stochastic process which turns out to be an equivalence of Brownian stochastic process by Section 5 turns out to be differentiable with respect to time. Indeed, an intriguing analytical feature of the path integral formulation is that it makes $x(t)$ differentiable everywhere with respect to time t, i.e., it makes $\dot{x}(t)$ a real continuous function of time t, from x_a at time t_a to x_b at time t_b by rewriting $x(t) = \bar{x}(t) + y(t)$ even with Brownian $dx(t)$, because the deterministic least-action path $\bar{x}(t)$ determined by EL equation is differentiable everywhere with respect to t from x_a to x_b. Meanwhile, the perturbed $y(t)$ with $y(t_a) = y(t_b) = 0$, without loss of generality, can be expressed in terms of Fourier series as

$$y(t) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi(t-t_a)}{t_b - t_a}\right) \tag{26}$$

with which coefficients a_n are random coefficients. Note that $y(t)$ written in the form of Fourier series above is a continuously differentiable function of t from x_a at time t_a to x_b at time t_b. As a result,

$$\begin{aligned}\dot{x}(t) &= \dot{\bar{x}}(t) + \dot{y}(t)\\ &= \dot{\bar{x}}(t) + \frac{\pi}{t_b - t_a}\sum_{n=1}^{\infty} n a_n \cos\left(\frac{n\pi(t-t_a)}{t_b - t_a}\right),\end{aligned} \tag{27}$$

which is obviously not only a continuous function of t but also differentiable with respect to t everywhere from x_a at time t_a to x_b at time t_b. From the perspective of making $x(t)$ differentiable with respect to t everywhere from x_a at time t_a to x_b at time t_b through path integral formulation by writing $x(t) = \bar{x}(t) + y(t)$ with $y(t_a) = y(t_b) = 0$ even with Brownian $dx(t)$, we break a ground for any further analytical exploration of the time path of $x(t)$ which is usually not differentiable everywhere with respect to t due to the Brownian process followed by $dx(t)$.

Since for both cases of zero inflation and time-varying inflation, the transition density of price gap $K(b|a)$ can be written in terms of $x(t) = \bar{x}(t) + y(t)$, the transition density formulations given above can thus be rewritten in terms of the least-action deterministic path $\bar{x}(t)$ from a to b and the perturbed path $y(t)$ from a to b. Also note that the path integrals treat the least-action deterministic path $\bar{x}(t)$ as a reference path or a constant path relative to the perturbed path $y(t)$ where fixing $y_a = y_b = 0$, it follows that the transition density $K(b|a) = K(x_b, t_b; x_a, t_a)$ above can be eventually written in the case of zero inflation with an implied state-dependent generalized hazard function $\Lambda(x)$ as

$$\begin{aligned} K^0(b|a) &= \int_{x_a}^{x_b} e^{-\frac{1}{\sigma^2}\int_{t_a}^{t_b}\left[\frac{1}{2}\dot{x}^2(\tau)+\sigma^2\Lambda(x)\right]d\tau}\mathcal{D}x(\tau) \\ &= e^{-\frac{1}{\sigma^2}S^0[\bar{x}(t)]}\int_0^0 e^{-\frac{1}{\sigma^2}S^0[y(t)]}\mathcal{D}y(t) \end{aligned} \tag{28}$$

and written in the presence of time-varying inflation with an implied state- and time-dependent generalized hazard function $\Lambda(x,t)$ as

$$\begin{aligned} K^{\mu(t)}(b|a) &= e^{\frac{\mu(t_b)}{\sigma^2}x_b}\int_{x_a}^{x_b} e^{-\frac{1}{\sigma^2}\int_{t_a}^{t_b}\left[\frac{1}{2}\dot{x}^2(\tau)+\sigma^2\Lambda(x,\tau)\right]d\tau}\mathcal{D}x(\tau) \\ &= e^{-\frac{1}{2\sigma^2}\int_{t_a}^{t_b}\mu^2(t)dt}e^{\frac{\mu(t_b)}{\sigma^2}x_b}e^{-\frac{1}{4\kappa}\int_{t_a}^{t_b}f^2(t)dt} \\ &\times e^{-\frac{1}{\sigma^2}S^{\mu(t)}[\bar{x}(t)]}\int_0^0 e^{-\frac{1}{\sigma^2}S^{\mu(t)}[y(t)]}\mathcal{D}y(t), \end{aligned} \tag{29}$$

where both the integrals above are taken with respect to the perturbed path $y(t)$ denoted by $\mathcal{D}y(t)$ rather than with respect to the ordinary integral usually denoted by dy in which y is not a stochastic path but an interval of real numbers. Hence, in the formulation of path integrals, we see the path integrals taken with respect to a stochastic (or perturbed) path $y(t)$ as $e^{-\frac{1}{\sigma^2}S[\bar{x}(t)]}\int_0^0 e^{-\frac{1}{\sigma^2}S[y(t)]}\mathcal{D}y(t)$ generally is not zero, because path integrals restricting two end points at zero just means the two end points, i.e., y_a and y_b, of the stochastic path $y(t)$ are fixed relative to the least-action deterministic reference path $\bar{x}(t)$. However, if all this is done in the sense of the ordinary integrals taken with respect to a real-valued interval in which y is any real number rather than a stochastic path $y(t)$, i.e., $\int_0^0 e^{-\frac{1}{\sigma^2}S[y]}dy$, then the result is always zero. In the proof of Appendix, we show how exactly the path integrals are performed with respect to a perturbed path $y(t)$ when fixing two end points of the perturbed path $y(t)$, y_a and y_b, relative to the least-action deterministic reference path $\bar{x}(t)$, i.e., $e^{-\frac{1}{\sigma^2}S[\bar{x}(t)]}\int_0^0 e^{-\frac{1}{\sigma^2}S[y(t)]}\mathcal{D}y(t)$.

We then can simply utilize the path integral formulation outlined above to analytically derive the transition density of the price gap in the context of sticky prices with quadratic generalized hazard function not only in the case of zero inflation but also in the presence of time-varying inflation. As an example, next proposition gives the analytical transition density of price gap in the context of sticky prices with quadratic generalized hazard function in the case of zero inflation.

Proposition 1. *The path integral transition density of state variable of price gap x, $K^0(x_b, t_b; x_a, t_a)$, or simply $K^0(b|a)$, going from x_a at time t_a to x_b at time t_b following a monetary shock that occurs at time $t = t_a$ to the sticky-price economy with an implied quadratic generalized hazard function $\Lambda(x) = \kappa x^2$ and volatility of the cost σ is given by*

$$K^0(b|a) = \left[\frac{\sqrt{2\kappa}}{2\pi\sigma \sinh\sqrt{2\kappa}\sigma(t_b - t_a)}\right]^{1/2} e^{-\frac{\sqrt{2\kappa}}{2\sigma}\left[\frac{\left(x_a^2+x_b^2\right)\cosh\sigma\sqrt{2\kappa}(t_b-t_a)-2x_a x_b}{\sinh\sigma\sqrt{2\kappa}(t_b-t_a)}\right]}. \tag{30}$$

Two observations here are worth noting. First, we show in the companion special theory paper using path integral transition density that with zero inflation the path integral transition density of sticky-price gap with firm's reinjection is equivalent to that without firm's reinjection. Hence, the transition density of price gap in the case of zero inflation as in Proposition 1 is the version that works for both cases. Second, from the path integral transition density of price gap in the case of zero inflation, $K^0(b|a)$, as in Proposition 1, we easily see that $K^0(b|a) = K^0(a|b)$, meaning the transition density of sticky-price gap in the case of zero inflation is symmetric. Later we will see this is no longer the case for the time-varying inflation scenario.

5 Path integral formulation in the presence of time-varying inflation

Remember that in the case of zero inflation the generalized hazard function takes the form of $\Lambda(x)$ which is only state-dependent, while when it comes to the time-varying inflation the generalized hazard function will take the functional form of $\Lambda(x, t)$ which is both state- and time-dependent. Now, consider a quadratic generalized hazard function as an example so that it keeps consistent with the first half of the paper. In the case of zero inflation as discussed in the first half of the paper, the generalized hazard function $\Lambda(x) = \kappa x^2$ which is state-dependent functional form, i.e., it is only dependent on the state variable, namely, the price gap x. By contrast, in the presence of time-varying inflation $\mu(t)$ which is generally a function of time t, the implied generalized hazard function is a function of both state variable x (i.e., the price gap) and time t. Furthermore, when it comes to the quadratic state- and time-dependent generalized hazard function implied by the time-varying inflation, it takes the form

$$\Lambda(x, t) = \kappa \left[x(t) + \frac{f(t)}{2\kappa} \right]^2. \tag{31}$$

Note that the state-dependent-only quadratic generalized hazard function in the case of zero inflation, $\Lambda(x) = \kappa x^2$, is just a special case of this more general state- and time-dependent quadratic generalized hazard function in the presence of time-varying inflation by noticing that $\Lambda(x) = \kappa x^2$ can be recovered by letting $f(t) = 0$. How to determine $f(t)$ via time-varying inflation $\mu(t)$ is a question that will be left for a future work in which the general equilibrium framework will be considered. Indeed, if the framework is not about general equilibrium as in this paper, there will be no determination of $f(t)$ via $\mu(t)$. The determination of $f(t)$ via time-varying inflation $\mu(t)$ only becomes possible only when the general equilibrium framework of the same sticky-price model is considered. In other words, once $f(t)$ as a function of $\mu(t)$ is determined in general equilibrium, the time-varying optimal return point $x^*(t)$ will also be determined in terms of the time-varying inflation by the equation $x^*(t) = -\frac{f(t)}{2\kappa}$ and hence the general equilibrium effect will be generated.

Transition density of price gap without firm's reinjection in the presence of time-varying inflation. The transition density of price gap following a shock going from x_a at time t_a to x_b at time t_b in the presence of time-varying inflation $\mu(t)$

without considering firm's reinjection is given by the following proposition as

Proposition 2. *The path integral transition density of price gap, $K^{\mu(t)}(x_b,t_b;x_a,t_a)$, following a monetary shock in the presence of time-varying inflation $\mu(t)$ without considering firm's reinjection, given generalized hazard function $\Lambda(x,t)=\kappa\big[x(t)+\frac{1}{2\kappa}f(t)\big]^2$ and cost volatility σ, is given by*

$$
\begin{aligned}
&K^{\mu(t)}(x_b,t_b;x_a,t_a)\\
&=\left[\frac{\sqrt{2\kappa}}{2\pi\sigma\sinh\sqrt{2\kappa}\sigma(t_b-t_a)}\right]^{1/2} e^{-\frac{1}{2\sigma^2}\int_{t_a}^{t_b}\mu^2(t)dt}e^{-\frac{2\mu(t_b)}{\sigma^2}x_b}e^{-\frac{1}{4\kappa}\int_{t_a}^{t_b}f^2(t)dt}\\
&\times e^{-\frac{\sqrt{2\kappa}}{2\sigma}\left[\frac{\left(x_a^2+x_b^2\right)\cosh\sigma\sqrt{2\kappa}(t_b-t_a)-2x_ax_b}{\sinh\sigma\sqrt{2\kappa}(t_b-t_a)}\right]}e^{-\frac{\sqrt{2\kappa}x_b}{2\sigma\sinh\sigma\sqrt{2\kappa}(t_b-t_a)}\int_{t_a}^{t_b}\left[-f(t)-\frac{\mu'(t)}{\sigma^2}\right]\sin\sigma\sqrt{2\kappa}(t-t_a)dt}\\
&\times e^{-\frac{\sqrt{2\kappa}x_a}{2\sigma\sinh\sigma\sqrt{2\kappa}(t_b-t_a)}\int_{t_a}^{t_b}\left[-f(t)-\frac{\mu'(t)}{\sigma^2}\right]\sin\sigma\sqrt{2\kappa}(t_b-t)dt}\\
&\times e^{\frac{\sqrt{2\kappa}}{2\sigma^3\kappa\sinh\sigma\sqrt{2\kappa}(t_b-t_a)}\int_{t_a}^{t_b}\int_{t_a}^{t}\left[-f(t)-\frac{\mu'(t)}{\sigma^2}\right]\left[-f(s)-\frac{\mu'(s)}{\sigma^2}\right]\sin\sigma\sqrt{2\kappa}(t_b-t)\sin\sigma\sqrt{2\kappa}(s-t_a)dsdt}.
\end{aligned}
\tag{32}
$$

Proof. See Appendix. □

We see that, from Proposition 2, the transition density of price gap in the presence of time-varying inflation even without firm's reinjection is no longer symmetric, that is, $K^{\mu(t)}(x|y)\neq K^{\mu(t)}(y|x)$. We will see this is still the case when the firm's reinjection is considered. Later we will also see that there only exists discrete times T^* at which the time-varying inflation converges to zero. If the monetary authority can precisely target these times T^* as the times at which the inflation target is achieved, then the whole economy in transition even with time-varying inflation will become symmetric again in terms of the transition density of the state variable of the price gap. We say achieving a symmetric economy in transition is what an effective and optimal monetary policy should aim for. By achieving a symmetric economy in transition in the presence of time-varying inflation effectively eliminates the negative impact that the time-varying inflation brings to the economy, and as a result the whole economy in transition even with time-varying inflation will behave more like an economy with zero inflation in transition. Indeed, we will see later in the paper that if the monetary authority can accurately set the discrete times T^* as the timing for the inflation target (for instance, zero inflation target for simplicity) to be achieved, then the marginal output impulse response in the presence of time-varying inflation will take the form of linear transformation of the marginal output impulse response with zero inflation, which is exactly why in this case the sticky-price economy even with time-varying inflation in transition following a monetary shock becomes symmetric again as in the zero-inflation economy in transition.

Reinjections of firms and transition density with time-varying inflation. Here, we discuss, in the presence of time-varying inflation, the transition density of price gap with firm's reinjection, $\mathcal{K}^{\mu(t)}(y|x)$, is not equal to the transition density of price gap without firm's reinjection, $K^{\mu(t)}(y|x)$, and they are related by the following proposition.

Proposition 3. *Let the path integral transition density of price gap with firm's reinjection in the presence of time-varying inflation be $\mathcal{K}^{\mu}(t)(y|x)$ and the path integral transition density of price gap without firm's reinjection in the presence of time-varying inflation be $K^{\mu}(t)(y|x)$, then $\mathcal{K}^{\mu}(t)(y|x) \neq K^{\mu}(t)(y|x)$, and they are related by*

$$\begin{aligned}
\mathcal{K}^{\mu(t)}(y|x) = K^{\mu(t)}(y|x) + e^{-\frac{1}{2\sigma^2}\int_0^t \mu^2(r)dr} e^{-\frac{1}{4\kappa}\int_0^t f^2(r)dr} \\
e^{\frac{\sqrt{2\kappa}}{2\sigma^3\kappa \sinh \sigma\sqrt{2\kappa}t}\int_0^t\int_0^r \left[f(r)+\frac{\mu'(r)}{\sigma^2}\right]\left[f(s)+\frac{\mu'(s)}{\sigma^2}\right]\sin\sigma\sqrt{2\kappa}(t-r)\sin\sigma\sqrt{2\kappa}s ds dr} \\
e^{-\frac{2\mu(t)}{\sigma^2}y+\frac{\sqrt{2\kappa}y}{2\sigma\sinh\sigma\sqrt{2\kappa}t}\int_0^t\left[f(r)+\frac{\mu'(r)}{\sigma^2}\right]\sin\sigma\sqrt{2\kappa}r dr} \\
e^{\frac{\sqrt{2\kappa}x}{2\sigma\sinh\sigma\sqrt{2\kappa}t}\int_0^t\left[f(r)+\frac{\mu'(r)}{\sigma^2}\right]\sin\sigma\sqrt{2\kappa}(t-r)dr} \\
\times\sum_{j=1}^{\infty}\Lambda(x^*(\tau))\phi_j(x^*(\tau))\int_0^t e^{\lambda_j(\tau-t)}d\tau\phi_j(x)\phi_j(y)
\end{aligned} \tag{33}$$

Proof. See Appendix. □

The analytical result obtained above implies that unlike the case of zero inflation, the transition density of price gap of a sticky-price economy in the presence of time-varying inflation with firm's reinjection is distinct from the transition density of price gap of a sticky-price economy in the presence of time-varying inflation without firm's reinjection. But the difference of the two becomes zero as time goes to infinity. In other words, as long as the the time horizon is strictly less than infinity, the transition density of price gap with firm's reinjection will be different from the transition density of price gap without firm's reinjection in the case of time-varying inflation.

6 The propagation of a monetary shock: an optimal monetary policy

6.1 General behavior of the propagation

Following discussion above, the relationship between the marginal output impulse response in the case of zero inflation $\mathcal{Y}^0(t)$ and the marginal output impulse response in the case of time-varying inflation with firm's reinjection $\mathcal{Y}^{\mu(t)}(t)$ can be established as in the next proposition.

Proposition 4. *The marginal output impulse response function following a monetary shock in the presence of time-varying inflation and an implied state- and time-dependent generalized hazard function with firm's reinjection can be rewritten as*

$$\begin{aligned}
&\mathcal{Y}^{\mu(t)}(t)\\
&= e^{-\frac{1}{2\sigma^2}\int_0^t \mu^2(r)dr} e^{-\frac{1}{4\kappa}\int_0^t f^2(r)dr}\\
&e^{\frac{\sqrt{2\kappa}}{2\sigma^3\kappa\sinh\sigma\sqrt{2\kappa}t}\int_0^t\int_0^r\left[f(r)+\frac{\mu'(r)}{\sigma^2}\right]\left[f(s)+\frac{\mu'(s)}{\sigma^2}\right]\sin\sigma\sqrt{2\kappa}(t-r)\sin\sigma\sqrt{2\kappa}s\,ds\,dr}\\
&\sum_{j=1}^{\infty} e^{-\lambda_j t}\left[\int_{-\infty}^{\infty}(-y)e^{-\left[\frac{2\mu(t)}{\sigma^2}-\frac{\sqrt{2\kappa}}{2\sigma\sinh\sigma\sqrt{2\kappa}t}\int_0^t\left[f(r)+\frac{\mu'(r)}{\sigma^2}\right]\sin\sigma\sqrt{2\kappa}r\,dr\right]y}\phi_j(y)dy\right]\times\\
&\left[\int_{-\infty}^{\infty} f'(x)e^{\left[\frac{\sqrt{2\kappa}}{2\sigma\sinh\sigma\sqrt{2\kappa}t}\int_0^t\left[f(r)+\frac{\mu'(r)}{\sigma^2}\right]\sin\sigma\sqrt{2\kappa}(t-r)dr\right]x}\phi_j(x)dx\right]\\
&+ e^{-\frac{1}{2\sigma^2}\int_0^t \mu^2(r)dr} e^{-\frac{1}{4\kappa}\int_0^t f^2(r)dr}\\
&e^{\frac{\sqrt{2\kappa}}{2\sigma^3\kappa\sinh\sigma\sqrt{2\kappa}t}\int_0^t\int_0^r\left[f(r)+\frac{\mu'(r)}{\sigma^2}\right]\left[f(s)+\frac{\mu'(s)}{\sigma^2}\right]\sin\sigma\sqrt{2\kappa}(t-r)\sin\sigma\sqrt{2\kappa}s\,ds\,dr}\\
&\sum_{j=1}^{\infty}\frac{\Lambda^*\phi_j^*}{\lambda_j}\left(1-e^{-\lambda_j t}\right)\left[\int_{-\infty}^{\infty}(-y)e^{-\left[\frac{2\mu(t)}{\sigma^2}-\frac{\sqrt{2\kappa}}{2\sigma\sinh\sigma\sqrt{2\kappa}t}\int_0^t\left[f(r)+\frac{\mu'(r)}{\sigma^2}\right]\sin\sigma\sqrt{2\kappa}r\,dr\right]y}\phi_j(y)dy\right]\times\\
&\left[\int_{-\infty}^{\infty} f'(x)e^{\left[\frac{\sqrt{2\kappa}}{2\sigma\sinh\sigma\sqrt{2\kappa}t}\int_0^t\left[f(r)+\frac{\mu'(r)}{\sigma^2}\right]\sin\sigma\sqrt{2\kappa}(t-r)dr\right]x}\phi_j(x)dx\right],
\end{aligned} \tag{34}$$

so that, as a special case, the marginal output impulse response function in the case of zero inflation written explicitly in terms of eigenvalue-eigenfunction decomposition form is given by replacing $\mu(t)$ everywhere above in $\mathcal{Y}^{\mu(t)}(t)$ by 0 for all t, that is,

$$\mathcal{Y}^0(t) = \sum_{j=1}^{\infty} e^{-\lambda_j t}\left[\int_{-\infty}^{\infty}(-y)\phi_j(y)dy\right]\left[\int_{-\infty}^{\infty} f'(x)\phi_j(x)dx\right], \tag{35}$$

where

$$\Lambda^* = \kappa[x^*(\tau)]^2 = \frac{1}{4\kappa} f^2(\tau),$$

$$\phi_j^* = \frac{1}{\pi^{1/4}(2^{j-1}(j-1)!)^{1/2}} \left(\frac{2\kappa}{\sigma^2}\right)^{1/8} H_{j-1}\left(-\left(\frac{2\kappa}{\sigma^2}\right)^{1/4} \frac{1}{2\kappa} f(\tau)\right) e^{-\left(\frac{\kappa}{2\sigma^2}\right)^{1/2} \frac{f^2(\tau)}{4\kappa^2}},$$

$$\lambda_j = \sigma\sqrt{2\kappa}\left(j - \frac{1}{2}\right),$$

$$\phi_j(x) = \frac{1}{\pi^{1/4}(2^{j-1}(j-1)!)^{1/2}} \left(\frac{2\kappa}{\sigma^2}\right)^{1/8} H_{j-1}\left(\left(\frac{2\kappa}{\sigma^2}\right)^{1/4} x\right) e^{-\left(\frac{\kappa}{2\sigma^2}\right)^{1/2} x^2},$$

and $f'(x)$ is first-order derivative of the stationary density of price gap.

6.2 Asymptotic behavior of the propagation

There is one specially important interesting aspect of the propagation of a monetary shock, and that is we are particularly interested in how the marginal output impulse response propagates following the monetary shock at the time $t = T$ where T denotes the time at which the time-varying inflation converges to zero, i.e., $\mu(T) = 0$. One important question to ask is that, would the propagation of the marginal output impulse response following a monetary shock in the presence of time-varying inflation, $\mathcal{Y}^{\mu(t)}(t)$, at time $t = T$, instantly approach to the propagation of the marginal output impulse response following a monetary shock in the case of zero inflation, $\mathcal{Y}^0(t)$, at time $t = T$? Or, would they approach to each other asymptotically at time $t = T$ as T increases ? (i.e., would they converge to each other asymptotically as T increases ?) Or, they would never approach to each other at all even asymptotically. (i.e., they would never converge to each other asymptotically even as T increases). We start to explore such an interesting question by giving the following proposition. Basically, Proposition 5 is about the following. $\mathcal{Y}^{\mu(t)}(T)$ takes the linear transformation form of $\mathcal{Y}^0(T)$ and hence they asymptotically approach to each other up to a linear transformation if and only if $T = T^* = \frac{n\pi}{\sigma\sqrt{2\kappa}}$ where $n = 1, 2, 3, ...$, σ is the cost volatility and κ is the curvature of the generalized hazard function. Otherwise, or equivalently, if $T \neq T^* = \frac{n\pi}{\sigma\sqrt{2\kappa}}$, $\mathcal{Y}^{\mu(t)}(T)$ and $\mathcal{Y}^0(T)$ would never approach to each other even asymptotically.

Proposition 5. *Given cost volatility of the firm σ and the rate of price adjustment $\Lambda(x)$ in the case of zero inflation as a quadratic function of price gap x, $\Lambda(x) = \kappa x^2$, the propagation of a monetary shock which is characterized by the marginal output impulse response in a sticky-price economy with time-varying inflation is a linear transformation, or linearly-transformed amplification, of the propagation of the marginal output impulse response in the sticky-price economy with zero inflation only at discrete times $T^* = \frac{n\pi}{\sigma\sqrt{2\kappa}}$ at which the inflation converges to zero in the economy of time-varying inflation, where $n = 1, 2, 3, ...$, i.e., $\mathcal{Y}^{\mu(t)}(T^*) = A(T^*) \times \mathcal{Y}^0(T^*) + B(T^*)$, where*

$$\mathcal{Y}^0(T^*) = \sum_{j=1}^{\infty} e^{-\lambda_j T^*} \left[\int_{-\infty}^{\infty} (-y)\phi_j(y)dy \right] \left[\int_{-\infty}^{\infty} f'(x)\phi_j(x)dx \right], \tag{36}$$

which is the corresponding propagation of the marginal output impulse response in the sticky-price economy with zero inflation at the discrete times T^* and $A(T^*)$ and $B(T^*)$ are given by

$$\begin{aligned} A(T^*) = {} & e^{-\frac{1}{2\sigma^2}\frac{T^*}{2}\sum_{n=1}^{\infty} a_n^2} e^{-\frac{1}{4\kappa}\frac{T^*}{2}\sum_{n=1}^{\infty} b_n^2} \\ & \times e^{\frac{\sqrt{2\kappa}}{2\sigma^3\kappa \sinh \sigma\sqrt{2\kappa}T^*} \int_0^{T^*} \int_0^t \left[f(t)+\frac{\mu'(t)}{\sigma^2}\right]\left[f(s)+\frac{\mu'(s)}{\sigma^2}\right] \sin\sigma\sqrt{2\kappa}(T^*-t) \sin\sigma\sqrt{2\kappa} s ds dt}, \end{aligned} \tag{37}$$

$$\begin{aligned} B(T^*) = {} & e^{-\frac{1}{2\sigma^2}\int_0^{T^*} \mu^2(r)dr} e^{-\frac{1}{4\kappa}\int_0^{T^*} f^2(r)dr} \\ & \times e^{\frac{\sqrt{2\kappa}}{2\sigma^3\kappa \sinh \sigma\sqrt{2\kappa}T^*} \int_0^{T^*} \int_0^t \left[f(t)+\frac{\mu'(t)}{\sigma^2}\right]\left[f(s)+\frac{\mu'(s)}{\sigma^2}\right] \sin\sigma\sqrt{2\kappa}(T^*-t) \sin\sigma\sqrt{2\kappa} s ds dt} \\ & \times \sum_{j=1}^{\infty} \frac{\Lambda^* \phi_j^*}{\lambda_j} \left(1 - e^{-\lambda_j T^*}\right) \left[\int_{-\infty}^{\infty} (-y)\phi_j(y)dy \right] \left[\int_{-\infty}^{\infty} f'(x)\phi_j(x)dx \right] \end{aligned} \tag{38}$$

where

$$\lambda_j = \sigma\sqrt{2\kappa}\left(j - \frac{1}{2}\right),$$

$$\phi_j(x) = \frac{1}{\pi^{1/4}(2^{j-1}(j-1)!)^{1/2}} \left(\frac{2\kappa}{\sigma^2}\right)^{1/8} H_{j-1}\left(\left(\frac{2\kappa}{\sigma^2}\right)^{1/4} x\right) e^{-\left(\frac{\kappa}{2\sigma^2}\right)^{1/2} x^2},$$

$f'(x)$ is first-order derivative of the stationary density of price gap, and a_n, b_n are the Fourier coefficients, and H_{j-1} is the Hermite polynomials of degree $j - 1$.

Proof. See Appendix. □

One important implication of Proposition 5 relies on its policy usefulness. As we see that there only exists discrete times $T^* = n\pi/\sigma\sqrt{2\kappa}$ at which the time-varying inflation converges to zero such that the propagation of marginal output impulse response following a monetary shock in the presence of time-varying inflation converges to the propagation of marginal output impulse response in the case of zero inflation at discrete times T^* in the sense that $\mathcal{Y}^{\mu(t)}(T^*) \sim A(T^*)\mathcal{Y}^0(T^*)$ with finite $A(T^*)$. For an effective monetary policy, it should make sure that the monetary policy can effectively drive the time-varying inflation to reach (zero) inflation target only at these discrete times $T^* = n\pi/\sigma\sqrt{2\kappa}$, where $n = 1, 2, 3, ...$

As already discussed before, there only exists discrete times $T^* = n\pi/\sigma\sqrt{2\kappa}$ with positive integer n for the monetary authority to set as the exact timing for the (zero) inflation target to be achieved by an optimal monetary policy, such that the whole sticky-price economy even with time-varying inflation in transition following a monetary shock from T^* and on behaves as if it were the transitional economy in the case of zero inflation in the sense that the economy with time-varying inflation in transition exhibits a symmetric property just as in the case of zero inflation and the new steady state of the economy in such a case would coincides with the initial steady state of the economy, which are both symmetric. The reason is because the marginal output impulse response with time-varying inflation in such a case takes the form of linear transformation of its counterpart in the case of zero inflation. In contrast, if the monetary authority is unable to accurately set the discrete times $T^* = n\pi/\sigma\sqrt{2\kappa}$ with positive integer n as the exact timing for the (zero) inflation target to be achieved, but instead sets some other timing $T \neq T^* = n\pi/\sigma\sqrt{2\kappa}$ as the times for the target, the whole economy in transition from T and on would not exhibit the symmetric property and the new steady state of the economy in terms of the distribution of the price gap following the monetary shock would be asymmetric, which is distinct from the symmetric initial steady state of the economy.

More intuitively, the reason why it is always optimal for monetary authority to set times $T^* = n\pi/\sigma\sqrt{2\kappa}$ as the times at which the (zero) inflation target is achieved is because the average speed of convergence of the marginal output impulse response following a monetary shock in the presence of time-varying inflation from 0 to time $T^* = n\pi/\sigma\sqrt{2\kappa}$, $\lim_{T^*\to\infty} \log \mathcal{Y}^{\mu(t)}(T^*)/T^*$, is equal to that of the marginal output impulse response following a monetary shock in the case of zero inflation from 0 to time $T^* = n\pi/\sigma\sqrt{2\kappa}$. Since the inflation converges to zero at times $T = T^* = n\pi/\sigma\sqrt{2\kappa}$, i.e., $\mu(T^*) = 0$, it follows that the average speed of convergence of the marginal output impulse response in the presence of time-varying inflation, $\mathcal{Y}^{\mu(t)}(t)$, will converge to that of the marginal output impulse response in the case of zero inflation after $T^* = n\pi/\sigma\sqrt{2\kappa}$ if and only if the average speed of convergence of the marginal output impulse response following a monetary shock in the presence of time-varying inflation from 0 to time $T^* = n\pi/\sigma\sqrt{2\kappa}$, $\lim_{T^*\to\infty} \log \mathcal{Y}^{\mu(t)}(T^*)/T^*$, is equal to that of the marginal output impulse response following a monetary shock in the case of zero inflation from 0 to time $T^* = n\pi/\sigma\sqrt{2\kappa}$.

We will see in the next section that whenever the inflation target is set to achieve at $T \neq T^* = n\pi/\sigma\sqrt{2\kappa}$, the average speed of convergence of the marginal output impulse response following a monetary shock in the presence of time-varying inflation from 0 to time $T \neq T^* = n\pi/\sigma\sqrt{2\kappa}$, $\lim_{T\to\infty} \log \mathcal{Y}^{\mu(t)}(T)/T$, is always faster than that of the marginal output impulse response following a monetary shock in the case of zero inflation from 0 to time T, which is not an effective monetary policy would like to see. This is because a faster average speed of convergence in the presence of time-varying inflation implies the response of the economy to the monetary shock during transition in the presence of time-varying inflation with $T \neq T^*$ is more sensitive, making the transitional state of the economy following a monetary shock more unstable.

However, it is always a difficult task for any monetary authority to accurately set times that equal to $T = T^* = n\pi/\sigma\sqrt{2\kappa}$ at which the time-varying inflation could be driven to zero by an effective monetary policy. Hence, in general, we would like to consider how and to what extend the propagation of a monetary shock at times $T \neq T^* = n\pi/\sigma\sqrt{2\kappa}$ at which the time-varying inflation converges to zero in a sticky-price economy with time-varying inflation differs from the propagation of a monetary shock at times $T = T^* = n\pi/\sigma\sqrt{2\kappa}$ at which the time-varying inflation converges to zero in the sticky-price economy with time-varying inflation. Section 7 of this paper will be mainly about this particular issue. We show that this question can properly be addressed based on the generalization of the path integral transition density of the price gap in its relation to the eigenvalue-eigenfunction transition density of the price gap in the presence of time-varying inflation using a path integral perturbation method not just for the quadratic form of state- and time-dependent generalized hazard function but also for any form of the state- and time-dependent generalized hazard function.

We will see that in the context of $\mu(0) = \mu(T) = 0$ (i.e., considering a time frame from 0 to finite T in which inflation is time-varying but equal to zero at two end points $t = 0$ and $t = T$), the monetary policy that results in $T = T^* = n\pi/\sigma\sqrt{2\kappa}$ with positive integer n corresponds to the situation where the eigenvalue-eigenfunction pair of the transition matrix of the economy at time 0 indexed by i, $(\lambda_i, \phi_i(x))$, shares the exactly same eigenvalue-eigenfunction pair of the transition matrix of the economy at time T which is also indexed by i, $(\lambda_i, \phi_i(y))$. By contrast, the monetary policy that results in $T \neq T^* = n\pi/\sigma\sqrt{2\kappa}$ with positive integer n corresponds to the situation where the eigenvalue-eigenfunction pair of the transition matrix of the economy at time 0 indexed by i, $(\lambda_i, \phi_i(x))$, is distinctly different from the eigenvalue-eigenfunction pair of the transition matrix of the economy at time T which is indexed by j, $(\lambda_j, \phi_j(y))$, where $j \neq i$.

7 Generalization of path integral formulation and eigenvalue-eigenfunction decomposition

This section focuses on how to relate path integral formulation of the transition density of price gap to the spectral (eigenvalue-eigenfunction) decomposition of the transition density of the price gap following a monetary shock. This section does not specify the functional form of the $\Lambda(x, t)$ as long as it is a function of both state x and time t, so that the analysis in this section of the paper will be of more importance to the wide range of macroeconomic topics. Specifically, we first introduce the path integral formulation of time-dependent perturbation to generalize the path integral formulation of transition density of price gap following a monetary shock in the presence of time-varying inflation in its relation to the spectral (eigenvalue-eigenfunction) decomposition formulation. Then, we use the analytical result of the generalization to explore how and to what extend the propagation of a monetary shock at times $T \neq T^* = n\pi/\sigma\sqrt{2\kappa}$ at which the time-varying inflation converges to zero in a sticky-price economy with time-varying inflation differs from the propagation of a monetary shock at times $T = T^* = n\pi/\sigma\sqrt{2\kappa}$ at which the time-varying inflation converges to zero in the sticky-price economy with time-varying inflation. We assume that the time-varying inflation $\mu(t)$ is considered over a time frame $[0, T]$ where $T < \infty$ and restrict the time-varying inflation $\mu(t)$ to be zero only at both end points, that is, $\mu(0) = \mu(T) = 0$. We then study the propagation of the transition dynamics of a sticky-price economy at time T.

7.1 The generalization of eigenvalue-eigenfunction transition density by path integral perturbation

The analytical process used for such an analysis is called "time-dependent perturbation of $\Lambda(x,t)$". Through such an analytical process, we show that the path integral transition density of the price gap, even in the presence of time-varying inflation with an implied state- and time-dependent generalized hazard function $\Lambda(x,t)$, can still be expressed in terms of the generalized eigenvalue-eigenfunction form. We begin our analysis with the following proposition.

Proposition 6. *Suppose the state- and time-dependent generalized hazard function in the presence of time-varying inflation, $\Lambda(x,t)$, can be written as $\Lambda(x)+\lambda(x,t)$, i.e., $\Lambda(x,t)=\Lambda(x)+\lambda(x,t)$, where $\Lambda(x)$ is the state-dependent generalized hazard function. For instance, in the presence of time-varying inflation, the implied quadratic state- and time-dependent generalized hazard function $\Lambda(x,t)$ can be approximated by the quadratic state-dependent generalized hazard function in the case of zero inflation $\Lambda(x)=\kappa x^2$ and time integral of $\lambda(x(t),t)$ over time interval $[0,T]$, i.e., $\int_0^T \lambda(x(t),t)dt$, is assumed to be small relative to the time integral of $\Lambda(x)$ over the time interval $[0,T]$, i.e., $\int_0^T \Lambda(x(t))dt$. That is, here, it is assumed that $\int_0^T \Lambda(x(t))dt$ is large but $\int_0^T \lambda(x(t),t)dt$ is small. Specifically, in our case of state- and time-dependent generalized hazard function implied by a time-varying inflation, $\Lambda(x,t)=\kappa\left[x(t)+\frac{f(t)}{2\kappa}\right]^2$, we take the unperturbed generalized hazard function taking state-dependent-only quadratic form $\Lambda(x)=\kappa x^2$ and the perturbed component $\lambda(x,t)=x(t)f(t)+f^2(t)/4\kappa$ and further assume that $\kappa\int_0^T x^2(t)dt$ is large but $\int_0^T[x(t)f(t)+f^2(t)/4\kappa]dt$ is small. Then, the transition density of price gap in the presence of time-varying inflation, $K^{\mu(t)}(y|x)$, can be rewritten in terms of the state-dependent generalized hazard function $\Lambda(x)$ as*

$$K^{\mu(t)}(y|x)=K^0_{\Lambda(x)}(y|x)+K^{(1)}_{\Lambda(x)}(y|x)+K^{(2)}_{\Lambda(x)}(y|x)+\cdots, \tag{39}$$

where

$$K^0_{\Lambda(x)}(y|x)=\int_x^y e^{-\frac{1}{\sigma^2}\int_0^T\left[\frac{1}{2}\dot{z}^2+\sigma^2\Lambda(z)\right]d\tau}\mathcal{D}z(\tau), \tag{40}$$

$$K^{(1)}_{\Lambda(x)}(y|x) = -\int_x^y e^{-\frac{1}{\sigma^2}\int_0^T \left[\frac{1}{2}\dot{z}^2+\sigma^2\Lambda(z)\right]d\tau} \int_0^T \Lambda(z(s))ds\mathcal{D}z(\tau), \tag{41}$$

$$K^{(2)}_{\Lambda(x)}(y|x) = \frac{1}{2}\int_x^y e^{-\frac{1}{\sigma^2}\int_0^T \left[\frac{1}{2}\dot{z}^2+\sigma^2\Lambda(z)\right]d\tau} \int_0^T \Lambda(z(s),s)ds \int_0^T \Lambda(z(r),r)dr\mathcal{D}z(\tau), \tag{42}$$

$$\cdots$$

Proof. See Appendix. □

The proposition above implies that in the case of time-varying inflation with an implied state- and time-dependent generalized hazard function $\Lambda(x,t)$, we can decompose $\Lambda(x,t)$ into two components, namely, state-dependent-only component $\Lambda(x)$ and state- and time-dependent component $\lambda(x,t)$, implying that the state- and time-dependent component $\lambda(x,t)$ can be viewed as the perturbed component around the unperturbed state-dependent but time-independent component $\Lambda(x)$.

7.2 The propagation at $t = T$ by generalized eigenvalue-eigenfunction decomposition

In this section, we study how and to what extend the propagation of a monetary shock at times $T \neq T^* = n\pi/\sigma\sqrt{2\kappa}$ at which the time-varying inflation converges to zero differs from the propagation of a monetary shock at times $T = T^* = n\pi/\sigma\sqrt{2\kappa}$ at which the time-varying inflation converges to zero in the sticky-price economy with time-varying inflation. The following proposition tells us the path integral transition density of price gap implies its corresponding (generalized) eigenvalue-eigenfunction form in the presence of time-varying inflation. Simply speaking, the following proposition is about the equivalence between path integral transition density of price gap and the (generalized) eigenvalue-eigenfunction transition density of the price gap in the presence of the time-varying inflation.

Proposition 7. *The propagation of the marginal output impulse response following a monetary shock in the presence of time-varying inflation at time T at which the time-varying inflation converges to zero or equivalently $\mu(T) = 0$, $\mathcal{Y}^{\mu(t)}(T)$, can be equivalently expressed in terms of generalized eigenvalue-eigenfunction transition density $\sum_{j=1}^{\infty}\sum_{i=1}^{\infty}\lambda_{ji}(T)\phi_j(y)\phi_i(x)$. That is,*

$$\mathcal{Y}^{\mu(t)}(T) = \sum_{j=1}^{\infty}\sum_{i=1}^{\infty}\lambda_{ji}(T)\left[\int_{-\infty}^{\infty}(-y)\phi_j(y)dy\right]\left[\int_{-\infty}^{\infty}f'(x)\phi_i(x)dx\right]. \tag{43}$$

As a special case, if $j = i$, or equivalently, if $T = T^* = n\pi/\sigma\sqrt{2\kappa}$ with positive integer n as implied by the optimal monetary policy, then the equivalence is reduced to

$$\mathcal{Y}^{\mu(t)}(T^*) = \sum_{j=1}^{\infty} \lambda_{jj}(T^*) \Big[\int_{-\infty}^{\infty} (-y)\phi_j(y)dy \Big] \Big[\int_{-\infty}^{\infty} f'(x)\phi_j(x)dx \Big], \tag{44}$$

where

$$\lambda_{jj}(T^*) = e^{-\lambda_j T^*} + \lambda_{jj}^{(1)}(T^*) + \lambda_{jj}^{(2)}(T^*) + \cdots,$$

$$\lambda_{jj}^{(1)}(T^*) = -\int_0^{T^*} \int_{-\infty}^{\infty} \phi_j(z)\Lambda(z,\tau)\phi_j(z)e^{-\lambda_j T^*} dz d\tau$$

$$= -e^{-\lambda_j T^*} \int_0^{T^*} \Lambda_{jj}(\tau)d\tau,$$

$$\lambda_{jj}^{(2)}(T^*) = \int_0^{T^*} \Big[\int_0^{\tau} \sum_{k=1}^{\infty} e^{-\lambda_j (T^*-\tau)} \Lambda_{jk}(\tau) e^{-\lambda_k(\tau - s)} \Lambda_{ki}(s) e^{-\lambda_i s} ds \Big] d\tau,$$

$$\lambda_{ji}(T) = \lambda_{ji}^{(1)}(T) + \lambda_{ji}^{(2)}(T) + \cdots,$$

$$\lambda_{ji}^{(1)}(T) = -\int_0^{T} \int_{-\infty}^{\infty} \phi_j(z)\Lambda(z,\tau)\phi_i(z)e^{-\lambda_j (T-\tau) - \lambda_i \tau} dz d\tau$$

$$= -e^{-\lambda_j T} \int_0^{T} \Lambda_{ji}(\tau)e^{(\lambda_j - \lambda_i)\tau} d\tau,$$

$$\lambda_{ji}^{(2)}(T) = \int_0^{T} \Big[\int_0^{\tau} \sum_{k=1}^{\infty} e^{-\lambda_j (T-\tau)} \Lambda_{jk}(\tau) e^{-\lambda_k(\tau - s)} \Lambda_{ki}(s) e^{-\lambda_i s} ds \Big] d\tau,$$

$$\cdots$$

and so forth, where $\Lambda_{jj}(\tau)$ is called the matrix element of generalized hazard function Λ between states j and j and defined as

$$\Lambda_{jj}(\tau) = \int_{-\infty}^{\infty} \phi_j(z)\Lambda(z,\tau)\phi_j(z)dz.$$

Similarly, Λ_{jk}, Λ_{ki}, and Λ_{ji} are defined as the matrix element of generalized hazard function Λ between states k and j, between states i and k, and between states i and j, respectively, as

$$\Lambda_{jk}(\tau) = \int_{-\infty}^{\infty} \phi_j(z)\Lambda(z,\tau)\phi_k(z)dz,$$

$$\Lambda_{ki}(\tau) = \int_{-\infty}^{\infty} \phi_k(z)\Lambda(z,\tau)\phi_i(z)dz,$$
$$\Lambda_{ji}(\tau) = \int_{-\infty}^{\infty} \phi_j(z)\Lambda(z,\tau)\phi_i(z)dz.$$

In the case of quadratic generalized hazard function, all the matrix elements of generalized hazard function Λ are written as

$$\Lambda_{jj} = \kappa \int_{-\infty}^{\infty} \phi_j(z)z^2\phi_j(z)dz,$$

$$\Lambda_{jk} = \kappa \int_{-\infty}^{\infty} \phi_j(z)z^2\phi_k(z)dz,$$

$$\Lambda_{ki} = \kappa \int_{-\infty}^{\infty} \phi_k(z)z^2\phi_i(z)dz,$$

$$\Lambda_{ji} = \kappa \int_{-\infty}^{\infty} \phi_j(z)z^2\phi_i(z)dz.$$

Proof. See Appendix. □

From Proposition 7, we see that the argument regarding the odd- versus even-indexed terms in the case of zero inflation as discussed earlier in this paper is also valid in the presence of time-varying inflation, simply because the transition density of price gap even in the presence of time-varying inflation can be expressed in the form of eigenvalue-eigenfunction decomposition. That is, the odd-indexed terms will also vanish in the analytical impulse response of output with the presence of time-varying inflation, thereby implying that the odd-indexed eigenvalues will also not exist in the analytical expression of the impulse response function of output even with the presence of time-varying inflation just as in the case of zero inflation.

Next, we can proceed to explicitly evaluate $\lambda_{ji}(T)$ and $\lambda_{jj}(T)$ to fully develop the equivalence. This paper will focus on the first-order approximation for the $\lambda_{ji}(T)$ and $\lambda_{jj}(T)$, i.e., we consider $\lambda_{jj}(T) = e^{-\lambda_j T} + \lambda_{jj}^{(1)}(T)$ and $\lambda_{ji}(T) = \delta_{ji}e^{-\lambda_i T} + \lambda_{ji}^{(1)}(T)$. The fully developed version of the equivalence is summarized in the following proposition.

Proposition 8. *The propagation of the marginal output impulse response following a monetary shock in the presence of time-varying inflation at time T at which the time-varying inflation converges to zero or $\mu(T) = 0$, $\mathcal{Y}^{\mu(t)}(T)$ without firm's reinjection,* *is expressed as*

$$\mathcal{Y}^{\mu(t)}(T) = \sum_{j=1}^{\infty}\sum_{i=1}^{\infty}\left[\kappa\left(\int_{-\infty}^{\infty}\phi_j(z)z^2\phi_i(z)dz\right)\frac{e^{-\lambda_j T}-e^{-\lambda_i T}}{\lambda_j-\lambda_i}\right]\left[\int_{-\infty}^{\infty}(-y)\phi_j(y)dy\right]\left[\int_{-\infty}^{\infty}f'(x)\phi_i(x)dx\right] \tag{45}$$

As a special case without firm's reinjection, if $j = i$, or equivalently, if $T = T^* = n\pi/\sigma\sqrt{2\kappa}$ with positive integer n as implied by the optimal monetary policy, the equivalence is reduced to

$$\mathcal{Y}^{\mu(t)}(T^*) = \sum_{j=1}^{\infty}\left[\left(1-\kappa T^*\int_{-\infty}^{\infty}\phi_j^2(z)z^2dz\right)e^{-\lambda_j T^*}\right]\left[\int_{-\infty}^{\infty}(-y)\phi_j(y)dy\right]\left[\int_{-\infty}^{\infty}f'(x)\phi_j(x)dx\right]. \tag{46}$$

With firm's reinjection, the corresponding propagation is given by

$$\begin{aligned}
&\mathcal{Y}^{\mu(t)}(T)\\
&=\sum_{j=1}^{\infty}\sum_{i=1}^{\infty}\left[\int_{-\infty}^{\infty}(-y)\phi_j(y)dy\right]\left[\int_{-\infty}^{\infty}f'(x)\phi_i(x)dx\right]\left[-\kappa\int_{-\infty}^{\infty}\phi_j(z)z^2\phi_i(z)dz\right]\\
&\times\left(\frac{e^{-\lambda_i T}-e^{-\lambda_j T}}{\lambda_j-\lambda_i}\right)\\
&+\sum_{j=1}^{\infty}\sum_{i=1}^{\infty}\left[\int_{-\infty}^{\infty}(-y)\phi_j(y)dy\right]\left[\int_{-\infty}^{\infty}f'(x)\phi_i(x)dx\right]\left[-\kappa\int_{-\infty}^{\infty}\phi_j(z)z^2\phi_i(z)dz\right]\\
&\times\left[\frac{\Lambda^*\phi_i^*}{\lambda_i}\left(\frac{1-e^{-\lambda_j T}}{\lambda_j}-\frac{e^{-\lambda_i T}-e^{-\lambda_j T}}{\lambda_j-\lambda_i}\right)\right]\\
&+\sum_{j=1}^{\infty}\sum_{i=1}^{\infty}\left[\int_{-\infty}^{\infty}(-y)\phi_j(y)dy\right]\left[\int_{-\infty}^{\infty}f'(x)\phi_i(x)dx\right]\left[-\kappa\int_{-\infty}^{\infty}\phi_j(z)z^2\phi_i(z)dz\right]\\
&\times\left[\frac{\Lambda^*\phi_j^*}{\lambda_j}\left(\frac{1-e^{-\lambda_i T}}{\lambda_i}-\frac{e^{-\lambda_i T}-e^{-\lambda_j T}}{\lambda_j-\lambda_i}\right)\right]\\
&+\sum_{j=1}^{\infty}\sum_{i=1}^{\infty}\left[\int_{-\infty}^{\infty}(-y)\phi_j(y)dy\right]\left[\int_{-\infty}^{\infty}f'(x)\phi_i(x)dx\right]\left[-\kappa\int_{-\infty}^{\infty}\phi_j(z)z^2\phi_i(z)dz\right]\\
&\times\left[\frac{(\Lambda^*)^2\phi_j^*\phi_i^*}{\lambda_j\lambda_i}\left(T-\frac{1-e^{-\lambda_i T}}{\lambda_i}-\frac{1-e^{-\lambda_j T}}{\lambda_j}+\frac{e^{-\lambda_i T}-e^{-\lambda_j T}}{\lambda_j-\lambda_i}\right)\right]
\end{aligned} \tag{47}$$

As a special case with firm's reinjection, if $j = i$, or equivalently, if $T = T^* = n\pi/\sigma\sqrt{2\kappa}$ with positive integer n as implied by the optimal monetary policy, the equiv-

alence is reduced to

$$
\begin{aligned}
&\mathcal{Y}^{\mu(t)}(T^*) \\
&= \sum_{j=1}^{\infty}\left[\int_{-\infty}^{\infty}(-y)\phi_j(y)dy\right]\left[\int_{-\infty}^{\infty}f'(x)\phi_j(x)dx\right]\left[-\kappa\int_{-\infty}^{\infty}\phi_j^2(z)z^2dz\right] \\
&\times\left[\frac{e^{-\lambda_j T^*}+\frac{\Lambda^*\phi_j^*}{\lambda_j}\left(1-e^{-\lambda_j T^*}\right)}{-\kappa\int_{-\infty}^{\infty}\phi_j^2(z)z^2dz}+T^*e^{-\lambda_j T^*}\right] \\
&+2\sum_{j=1}^{\infty}\left[\int_{-\infty}^{\infty}(-y)\phi_j(y)dy\right]\left[\int_{-\infty}^{\infty}f'(x)\phi_j(x)dx\right]\left[-\kappa\int_{-\infty}^{\infty}\phi_j^2(z)z^2dz\right] \\
&\times\left[\frac{\Lambda^*\phi_j^*}{\lambda_j}\left(\frac{1-e^{-\lambda_j T^*}}{\lambda_j}-T^*e^{-\lambda_j T^*}\right)\right] \\
&+\sum_{j=1}^{\infty}\left[\int_{-\infty}^{\infty}(-y)\phi_j(y)dy\right]\left[\int_{-\infty}^{\infty}f'(x)\phi_j(x)dx\right]\left[-\kappa\int_{-\infty}^{\infty}\phi_j^2(z)z^2dz\right] \\
&\times\left[\frac{(\Lambda^*)^2(\phi_j^*)^2}{(\lambda_j)^2}\left(T^*-2\frac{1-e^{-\lambda_j T^*}}{\lambda_j}+T^*e^{-\lambda_j T^*}\right)\right]
\end{aligned}
\tag{48}
$$

where

$$\lambda_j=\sigma\sqrt{2\kappa}\left(j-\frac{1}{2}\right),$$

$$\phi_j(x)=\frac{1}{\pi^{1/4}(2^{j-1}(j-1)!)^{1/2}}\left(\frac{2\kappa}{\sigma^2}\right)^{1/8}H_{j-1}\left(\left(\frac{2\kappa}{\sigma^2}\right)^{1/4}x\right)e^{-\left(\frac{\kappa}{2\sigma^2}\right)^{1/2}x^2},$$

$$\lambda_i=\sigma\sqrt{2\kappa}\left(i-\frac{1}{2}\right),$$

$$\phi_i(x)=\frac{1}{\pi^{1/4}(2^{i-1}(i-1)!)^{1/2}}\left(\frac{2\kappa}{\sigma^2}\right)^{1/8}H_{i-1}\left(\left(\frac{2\kappa}{\sigma^2}\right)^{1/4}x\right)e^{-\left(\frac{\kappa}{2\sigma^2}\right)^{1/2}x^2},$$

and $f'(x)$ is first-order derivative of the stationary density of price gap, $H_{j-1}(\cdot)$ and $H_{i-1}(\cdot)$ are the Hermite polynomials of degree $j-1$ and $i-1$, respectively, where $i=1,2,3,...,$ and $j=1,2,3,....$

Finally, based on the analytical results obtained in Proposition 8 above, we conclude our discussion for the paper with an analytical result with regard to the three important average speeds of convergence, so that we can easily see, assuming zero inflation is the long-run inflation target set by the monetary authority, which scenario is more consistent with the optimal timing for achieving (zero) inflation target by an effective monetary policy and which scenario is less consistent with the optimal timing for achieving (zero) inflation target. In the next proposition, we give average speed of convergence of marginal output impulse response following a monetary shock in the presence of time-varying inflation from 0 to time $T \neq T^* = n\pi/\sigma\sqrt{2\kappa}$, $\lim_{T\to\infty} \log \mathcal{Y}^{\mu(t)}(T)/T$, average speed of convergence of marginal output impulse response following a monetary shock in the presence of time-varying inflation from 0 to time $T = T^* = n\pi/\sigma\sqrt{2\kappa}$, $\lim_{T^*\to\infty} \log \mathcal{Y}^{\mu(t)}(T^*)/T^*$, and average speed of convergence of marginal output impulse response following a monetary shock in the case of zero inflation from 0 to time T, $\lim_{T\to\infty} \log \mathcal{Y}^{0}(T)/T$.

Proposition 9. *Assuming quadratic state- and time-dependent generalized hazard function with curvature κ and the cost volatility σ, the average speed of convergence of marginal output impulse response following a monetary shock in the presence of time-varying inflation from 0 to time $T \neq T^* = n\pi/\sigma\sqrt{2\kappa}$, $\lim_{T\to\infty} \log \mathcal{Y}^{\mu(t)}(T)/T$, is given by*

$$\lim_{T\to\infty} \log \mathcal{Y}^{\mu(t)}(T)/T = \lambda_4 - \lambda_2 = 2\sigma\sqrt{2\kappa}. \tag{49}$$

The average speed of convergence of marginal output impulse response following a monetary shock in the presence of time-varying inflation from 0 to time $T = T^* = n\pi/\sigma\sqrt{2\kappa}$, $\lim_{T^*\to\infty} \log \mathcal{Y}^{\mu(t)}(T^*)/T^*$, is given by

$$\lim_{T^*\to\infty} \log \mathcal{Y}^{\mu(t)}(T^*)/T^* = \lambda_2 = \frac{3}{2}\sigma\sqrt{2\kappa}. \tag{50}$$

And average speed of convergence of marginal output impulse response following a monetary shock in the case of zero inflation from 0 to time T, $\lim_{T\to\infty} \log \mathcal{Y}^{0}(T)/T$, is given by

$$\lim_{T\to\infty} \log \mathcal{Y}^{0}(T)/T = \lambda_2 = \frac{3}{2}\sigma\sqrt{2\kappa}. \tag{51}$$

It is not surprising that the average speed of convergence of marginal output impulse response following a monetary shock in the presence of time-varying inflation from 0 to time $T = T^* = n\pi/\sigma\sqrt{2\kappa}$, $\lim_{T^*\to\infty} \log \mathcal{Y}^{\mu(t)}(T^*)/T^*$, coincides with the average speed of convergence of marginal output impulse response following a monetary shock in the case of zero inflation from 0 to time T, $\lim_{T\to\infty} \log \mathcal{Y}^{0}(T)/T$ and both of them are equal to λ_2, due to the fact that the former one takes the linear transformation of the latter one, as already discussed earlier in this paper. The reason why the average speed of convergence of marginal output impulse response following a monetary shock in the presence of time-varying inflation from 0 to time $T \neq T^* = n\pi/\sigma\sqrt{2\kappa}$, $\lim_{T\to\infty} \log \mathcal{Y}^{\mu(t)}(T)/T$ is $\lambda_4 - \lambda_2$ which is greater than λ_2 is completely due to the analytical functional form of the $\mathcal{Y}^{\mu(t)}(T)$ with $T \neq T^*$ as stated in Proposition 12. Here, we see the average speed of convergence of marginal output impulse response following a monetary shock in the presence of time-varying inflation from 0 to time $T \neq T^* = n\pi/\sigma\sqrt{2\kappa}$ is greater than that of marginal output impulse response in the presence of time-varying inflation from 0 to time $T^* = n\pi/\sigma\sqrt{2\kappa}$ and thus an optimal monetary policy should avoid setting $T \neq T^* = n\pi/\sigma\sqrt{2\kappa}$ as the times at which the inflation target is achieved but rather should wisely choose $T = T^* = n\pi/\sigma\sqrt{2\kappa}$ as the optimal timing for achieving the long-run inflation target.

8 Conclusion

This paper studies macroeconomic dynamics in a sticky price setting using path integrals. We analytically explored the transition dynamics of the economy with time-varying inflation in sticky- price setting. The technique used for our analysis, path integral formulation, turns out to be the most important mathematical contribution of this paper which provides us an ideal mathematical tool that enables us to explore many unsolved challenges in sticky price framework analytically. This paper aims to help lay out the theoretical foundations for the topic and therefore also aims to leave all other practical considerations like quantitative analysis for a future work. Even without any of such quantitative or empirical aspect of the analysis, we still view the work as groundbreaking and transformative in a way that the theoretical foundation itself and the optimal monetary policy analysis alongside introducing path integrals into macroeconomics is important enough for this work to be an independent and promising work that could potentially lead to more future explorations of macroeconomic dynamics studies based on path integral formulation.

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