Special Theory of Sticky Prices with Path Integrals

Mingyi Yang*

Washington State University

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Abstract

This paper introduces path integrals as a mathematical technique into macroeconomics to study the dynamics of sticky prices with generalized hazard functions following a monetary shock. So far, the only existing analytical technique that can be applied to study the dynamics of sticky prices with generalized hazard functions is the spectral (eigenvalue-eigenfunction) decomposition technique, which was taken by Alvarez and Lippi (2022). The most obvious advantage of path integral formulation in studying sticky-price model compared to the spectral (eigenvalue-eigenfunction) decomposition technique as used in Alvarez and Lippi (2022) is that the path integral transition density of price gap turns out to be not in the form of eigenvalue-eigenfunction decomposition, but the path integral transition density of the price gap implies and can lead to that eigenvalue-eigenfunction decomposition. That is, we see path integral formulation as a more general and advanced mathematical technique compared to the eigenvalue-eigenfunction decomposition approach.

JEL Classification Numbers: E3, E5

Key Words: sticky prices, generalized hazard functions, path integrals, monetary shock, transition density, eigenvalue-eigenfunction decomposition.

1 Introduction

At the beginning of the paper, we must state that the words chosen in the title of the paper "Special Theory" aims to suggest that this paper is to study the dynamics of

*School of Economic Sciences, Washington State University in Pullman. mingyi.yang2@wsu.edu.

sticky prices in an economy with zero inflation. In a companion paper titled "General Theory of Sticky Prices and Optimal Monetary Policy with Path Integrals", we study the dynamics of sticky prices in an economy with time-varying inflation. We adapt this "special-general" theory tradition from the famous "special-general" theory of relativity proposed by Albert Einstein in which the special theory of relativity applies when the gravity is not considered, while the general theory of relativity applies when the gravity cannot be ignored. In other words, we analogously compare the (time-varying) inflation in our sticky-price framework to the gravity in relativity theory.

This paper introduces path integrals as a mathematical technique into macroeconomics to study the dynamics of sticky prices with generalized hazard functions following a monetary shock. So far, the only existing analytical technique that can be applied to study the dynamics of sticky prices with generalized hazard functions is the spectral (eigenvalue-eigenfunction) decomposition technique, which was taken by Alvarez and Lippi (2022). We should briefly talk about what spectral decomposition means in the context of the sticky-model model with which the price gap between the logarithmic charging price of the firm and the logarithmic optimal price that maximizes the firm's profit is the state variable. The basic idea of the spectral decomposition from Alvarez and Lippi (2022) is to obtain the eigenvalues and their corresponding eigenfunctions of the transition matrix $\mathcal{A}^*$, a differential operator with respect to the state variable that appears on the right hand side of the time-dependent Kolmogorov Forward Equation (KFE) and the solution to the time-dependent KFE gives the time evolution of the density of the state variable of the model. With the help of the obtained eigenvalues and their corresponding eigenfunctions of the transition matrix $\mathcal{A}^*$, both the analytical solution to the time-dependent KFE as well as the transition density of the state variable take the form of eigenvalue-eigenfunction decomposition. The fact that both the time evolution of the density of price gap and the transition density of the price gap take eigenvalue-eigenfunction decomposition form makes it very convenient for us to analytically study the marginal impulse response function of the output in such a context with the eigenvalues generally denoting the average speed of the convergence of the transition process of the output.

This paper instead introduces a new mathematical technique learned from theoretical physics called "Path Integrals" to analytically study the dynamics of sticky prices with generalized hazard functions following a monetary shock. Note that the original version of path integrals in theoretical physics is a complex-version, which does not fit to our sticky-price framework. Thus, the economic-friendly version of path integrals that is being used in this paper to study the dynamics of sticky prices is the real-version which is the revised version of the original path integrals in physics. The most obvious advantage of path integral formulation in studying sticky-price model compared to the spectral (eigenvalue-eigenfunction) decomposition technique as used in Alvarez and Lippi (2022) is that the path integral transition density of price gap turns out to be not in the form of eigenvalue-eigenfunction decomposition, but the path integral transition density of the price gap implies and can lead to that eigenvalue-eigenfunction decomposition. In contrast, the eigenvalue-eigenfunction decomposition approach does not imply and hence cannot lead to its corresponding path integral formulation. In other words, we can clearly see here that the path integral formulation is a more advanced and general analytical technique in studying the sticky-price model with generalized hazard functions compared to the existing spectral (eigenvalue-eigenfunction) decomposition approach as used in Alvarez and Lippi (2022). In a companion paper studying the dynamics of sticky prices in the presence of time-varying inflation, we prove that

The setup. The uncontrolled stochastic price (gap) process following the monetary shock for the firm is given by

$$dx(t) = \sigma dW(t). \tag{1}$$

By uncontrolled price process, we mean this is the price (gap) process in the absence of price adjustment which only happens at the stopping time(s), τ(s). Here, $x(t)$ is our idiosyncratic state which is called price gap measuring the difference of log-transformed price currently charged by the firm and the optimal price that maximizes firm's profit. In this paper, we assume $x(t) \in (-\infty, \infty)$. Note that this paper studies special theory of sticky prices with which the economy is drift-less. Or equivalently, it is a zero-inflation economy that is associated with the special theory of sticky prices. And therefore, there is no drift in the uncontrolled price process in this paper. In a companion paper, we study the general theory of sticky prices where we will have time-varying drift (inflation) added as a part of the uncontrolled price process.

The setup for firm's problem with zero inflation is a quite standard economic environment which has been extensively studied by Nakamura and Steinsson (2010), Woodford (2009), Costain and Nakov (2011), Caballero and Engel (1999), Alvarez and

Lippi (2022), Alvarez, Lippi, and Oskolkov (2022), Caplin and Spulber (1987), Caplin and Leahy (1991), and Bils and Klenow (2004). When it comes to price adjustment at stopping time τ, the firm first exits the distribution at the rate of price adjustment given by the generalized hazard function and then re-enters the distribution or is said to be reinjected into the distribution at the optimal return point x^* that maximizes firm's profit, and as a result, the price gap x becomes zero for those firms that have just reset their prices. This paper will explicitly show that, in the case of special theory of sticky prices with zero inflation, the transition dynamics of the sticky-price economy following a monetary shock with generalized hazard functions and with reinjection of the firm is equivalent to the transition dynamics of the economy without considering reinjection of the firm. Therefore, in the case of zero inflation, we only need to keep track of the firm until the stopping time τ at which the firm resets its prices and simply ignore the reinjection of the firm afterwards.

This paper uses generalized hazard functions $\Lambda(x)$ to characterize all essential sticky-price features with the special theory. Generalized hazard functions were originally developed by Caballero and Engel (1993a) and Caballero and Engel (1993b), Dotsey, King, and Wolman (1999) and further studied by Caballero and Engel (1999), Woodford (2009) and Costain and Nakov (2011). Generalized hazard functions have also been recently studied by Alvarez and Lippi (2022) and Alvarez, Lippi, and Oskolkov (2022). In general, generalized hazard function $\Lambda(x)$ is a function: $x \to \mathbb{R}^+ \cup \{0\}$, that maps the idiosyncratic state, i.e., the price gap x, to the rate of the price adjustment over time. Clearly, it requires $\Lambda(x^*) = 0$ because the optimal return point $x^*(\tau)$ closes up the price gap, i.e., price gap $x = 0$ at x^*. Since zero price gap contributes zero incentive for the firm to change its price, it follows that the rate of price adjustment $\Lambda(x)$ is zero at $x = x^*$. With zero inflation, i.e., $\mu = 0$, the optimal return point is $x^* = 0$.

The differential operator approach. Given the setup, the corresponding time-dependent KFE characterzing the time evolution of density of price gap, $p(x,t)$, following a monetary shock, is written as

$$p_t(x,t) = \frac{\sigma^2}{2} p_{xx}(x,t) - \Lambda(x)p(x,t) + \Lambda(x)\delta(x), \tag{2}$$

where $\delta(x)$ is the Dirac delta function centered about $x = 0$ and the term $\Lambda(x)\delta(x)$ aims to account for the reinjection of the firm at $x^* = 0$ right after the price adjustment in the case of special theory with zero inflation. Later in this paper we show that in the case of special theory of sticky prices with zero inflation, the path integral transition density of price gap is equivalent regardless of whether the reinjection of the firm is considered or not. Here, we just use that result beforehand, so that the KFE above for studying the sticky-price model with the special theory can be equivalently reduced to

$$p_t(x,t) = \frac{\sigma^2}{2} p_{xx}(x,t) - \Lambda(x) p(x,t). \tag{3}$$

Now, we define a differential operator $\mathcal{B}$ as

$$\mathcal{B}p = \frac{\sigma^2}{2} p_{xx} - \Lambda(x) p, \tag{4}$$

then the time-dependent KFE can be rewritten in terms of the differential operator as

$$p_t = \mathcal{B}p, \tag{5}$$

and the eigenvalues λ and the corresponding eigenfunctions (i.e., the eigenvectors) ϕ of transition matrix $\mathcal{B}$ (by definition of the eigenvalues and eigenvectors of a matrix in matrix algebra) are determined by

$$\mathcal{B}\phi = \lambda\phi. \tag{6}$$

Here, the (formal) adjoint of $\mathcal{B}$ is the operator $\mathcal{B}^*$ which is equal to $\mathcal{B}$ as long as there is no drift μ, i.e., here, we have $\mathcal{B} = \mathcal{B}^*$ with $\mu = 0$. We say any operator $\mathcal{B}$ satisfying $\mathcal{B} = \mathcal{B}^*$ is a self-adjoint operator. Hence, in the special theory of sticky prices with zero inflation, we always have that $\mathcal{B}$ is self-adjoint. It is important because it is well known that eigenvalues of a self-adjoint operator are real. Not every operator in the domain of sticky-price model is self-adjoint and hence has real-valued eigenvalues. For instance, in a companion paper with time-varying inflation, the resulting differential operator $\mathcal{A}$ is defined as

$$\mathcal{A}u = \mu(t) u_x + \frac{\sigma^2}{2} u_{xx} - \Lambda(x,t) u \tag{7}$$

whose adjoint is given by

$$\mathcal{A}^{*}p = -\mu(t)p_x + \frac{\sigma^2}{2}p_{xx} - \Lambda(x,t)p. \tag{8}$$

Obviously, $\mathcal{A} \neq \mathcal{A}^{*}$ and hence the differential operators in the case of general theory of sticky prices with time-varying inflation, $\mathcal{A}$ and $\mathcal{A}^{*}$, are not self-adjoint and therefore the eigenvalues of both transition matrices $\mathcal{A}$ and $\mathcal{A}^{*}$ can be complex. So, in the general theory of sticky prices with time-varying inflation, we must transform operator $\mathcal{A}$ which is not self-adjoint into a self-adjoint operator $\mathcal{B}$ before we can proceed to conduct any meaningful analysis there. In the companion paper, we show that the differential operators $\mathcal{A}$ and $\mathcal{B}$ are closely related and can be easily transformed between one and another.

The path integral formulation. What this paper will do is to show that there exists a more general mathematical technique compared to the operator-based KFE formulation based on which the eigenvalue-eigenfunction decompostion of the transition matrix is obtained, path integral formulation, that implies and also can efficiently lead us to the corresponding eigenvalue-eigenfunction decomposition of the transition matrix. In particular, first, we will show that the path integral formulation implies its corresponding time-dependent KFE and hence the operator-based transition matrix. Second, we will show that the path integral transition density implies its corresponding eigenvalue-eigenfunction decomposition of the transition matrix. Finally, we will show that, in a special case with quadratic generalized hazard function, the path integral transition density enables us to obtain the corresponding analytical eigenvalues and eigenfunctions of the transition matrix with quadratic generalized hazard function. Above all, this paper breaks a ground for us to be able to study the dynamics of sticky prices by using a path integral based formulation as opposed to the differential operator based approach, i.e., it is about the path integral versus differential operator.

A monetary shock. This paper aims to explore the dynamics of a sticky-price economy with generalized hazard functions following a monetary shock. In terms of a monetary shock studied by this paper, we consider a parallel shift in all price gaps. The rationale is that, under the specific assumptions this paper follows, the parallel shift in the level of money supply maps into a parallel shift in nominal wages and thus parallel shift in all price gaps. When it comes to uncertainty shocks, however, we do not have the parallel shift in all price gaps anymore. Specifically, with uncertainty shocks, the dispersion of the initial steady-state distribution of the price gaps will be changed without a parallel shift in distribution of the price gaps. Furthermore, in a sequence of uncertainty and monetary shocks, we will have the mixture of both parallel shift in the price gaps and the changes in the dispersion of the price gaps.

Specifically, when it comes to the distributional dynamics of the price gap following a monetary shock, we take a similar approach as in Alvarez, Lippi, and Souganidis (2023); that is, we consider a perturbation ν of the stationary density of price gap $f(x)$, or equivalently, we define the initial condition of the density of price gap right after the monetary shock of size δ, $f_0(x)$, as

$$f_0(x) = f(x) + \delta\nu(x), \tag{9}$$

where $\int_{-\infty}^{\infty} \nu(x)dx = 0$.

In the spirit of Alvarez, Lippi, and Souganidis (2023) and in the context of small monetary shock characterized by the small size of the monetary shock δ (i.e., small δ), there is a particular perturbation focused by this paper which is the one corresponding to an unanticipated aggregate nominal shock that changes the nominal costs of all firms by an amount δ, so that the initial condition for the density of price gap before any decision is taken is

$$f_0(x) = f(x + \delta), \tag{10}$$

which is a special case of $f_0(x) = f(x)+\delta\nu(x)$ where $\nu(x) = f'(x)$, which follows from the fact that $f(x+\delta) = f(x)+\delta f'(x)+\mathcal{O}(\delta)$ implied by Taylor expansion of $f(x+\delta)$. The interpretation of such an initial condition of the density of price gap is that after the monetary shock of size δ the nominal cost jumps immediately and hence the value of the price gap x for each firm jumps from x to $x - \delta$. Hence, in this paper, the signed measure $\hat{f}(x) = f(x+\delta)-f(x)$ describing the deviation of the initial condition of the density of price gap from the stationary density of price gap right after the monetary shock to the stationary density is given by $\hat{f}(x) = \delta f'(x) + \mathcal{O}(\delta)$. Note that the impulse response function is basically the expected value of any variable of interest computed on this signed measure and that is why this signed measure is an important component of the impulse response function of any variable of our concern.

For the consideration of the marginal version of the monetary shock, we thus have

$$\left.\frac{\partial \hat{f}(x)}{\partial \delta}\right|_{\delta\to 0} = f'(x), \tag{11}$$

which is the version of the signed measure that will be used in the marginal output impulse response function of this paper.

Impulse response function. We use output impulse response function

$$Y(t;\delta) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(-y)q_t(y|x)dyd\hat{F}(x), \tag{12}$$

where $F(x)$ is the corresponding cumulative density and thus the corresponding marginal version of output impulse response function as $\delta \to 0$, $\mathcal{M}(t)$, is given by

$$\mathcal{M}(t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(-y)q_t(y|x)f'(x)dxdy, \tag{13}$$

where $q_t(y|x)$ is the transition density of price gap from x at time 0 to y at time t and $-y$ is due to the fact that output is inversely proportional to output gap.

2 Introduction to path integral formulation with generalized hazard functions

This section introduces path integrals in the context of sticky prices and generalized hazard functions to explore the distributional dynamics of the sticky-price economy following a monetary shock in the case of both zero inflation and time-varying inflation. The theoretical framework of path integrals outlined here generally works not only for an economy of sticky prices in the case of zero inflation with an implied state-dependent-only generalized hazard function but also for a sticky-price economy in the presence of time-varying inflation with an implied state- and time-dependent generalized hazard function. The only difference between the two cases is that the generalized hazard function in the former case of zero inflation takes the form of $\Lambda(x)$ and the generalized hazard function in the latter case of time-varying inflation takes the form of $\Lambda(x,t)$.

The fundamental idea about path integrals (or path integral formulation) for studying macroeconomic dynamics of sticky prices with generalized hazard functions following a monetary shock is to analytically obtain the transition density of price gap x going from x_a at time t_a to x_b at time t_b, $K(b|a)$, or simply, the transition probability that price gap ends up being x_b at time t_b given it starts with x_a at time t_a. Now, imagine the following thought experiment. (Also see page 59 of Feynman and Hibbs (1965), i.e., the Gaussian Integrals section). First, following a shock, let us denote a deterministic time path of price gap x from x_a at time t_a to x_b at time t_b by $\bar{x}(t)$ which is the deterministic time path of x based on the principle of least action, and the actual time path of the price gap x from t_a to t_b by $x(t)$, where $t \in [t_a, t_b]$. Then, the actual time path of price gap over the transition period $x(t)$ can be written as the sum of the deterministic time path $\bar{x}(t)$ of least action and the deviation of the actual time path $x(t)$ from the deterministic path $\bar{x}(t)$ of least action, namely, $y(t)$, as

$$x(t) = \bar{x}(t) + y(t) \tag{14}$$

that is, instead of defining a point on the path by its distance $x(t)$, we measure instead the deviation $y(t)$ from the least-action deterministic path $\bar{x}(t)$. Given the transitional time path from t_a to t_b, both the actual and the least-action deterministic time path of price gap from the transition period t_a to t_b following a shock have the same initial and terminal locations because they are actually both the transitional time paths between x_a at time t_a and x_b at time t_b (i.e., fixing end points but varying the path in-between), that is,

$$x(t_a) = \bar{x}(t_a) = x_a$$

and

$$x(t_b) = \bar{x}(t_b) = x_b,$$

and therefore, based on $x(t) = \bar{x}(t) + y(t)$, we get

$$y(t_a) = y(t_b) = 0,$$

that is, the deviation of the actual time path of price gap from the least-action deterministic time path of the price gap at the initial and terminal locations x_a and x_b, respectively, are both equal to zero.

In between these end points $y(t)$ can take any form. Since the least-action deterministic path $\bar{x}(t)$ is non-random and can always be solvable according to the principle of least action in which Euler-Lagrange (EL) equation applies, any variation by a perturbation in the alternative path $x(t)$ is equivalent to the associated variation in $y(t)$. Thus, in a path integral, the path differential $\mathcal{D}x(t)$ can be replaced by $\mathcal{D}y(t)$, i.e., $\mathcal{D}x(t) = \mathcal{D}y(t)$, and the path $x(t)$ by $\bar{x}(t) + y(t)$. Here, we use $\mathcal{D}$ to denote path differential rather than the ordinary differential d used in the standard calculus.

In this form, $\bar{x}(t)$ is the least-action deterministic path for the integration which is analytically given by EL equation. Moreover, the stochastic path $y(t)$ is restricted to take the value 0 at both end points. This substitution leads to a path integral independent of end-point positions. See Page 59 of Feynman and Hibbs (1965). In what follows, we specifically illustrate how to use path integral formulation to analytically explore the transition dynamics of a sticky-price economy following a monetary shock for the cases where zero inflation $\mu(t) = 0$ implying state-dependent generalized hazard function $\Lambda(x)$ and time-varying inflation $\mu(t)$ implying state- and time-dependent generalized hazard function $\Lambda(x,t)$ associated with the volatility of the economy, σ.

It follows from the definition of the path integral formulation Feynman and Hibbs (1965) that the path integrals, given our economic settings, are formulated by the following integral for the transition density, $K(x_b, t_b; x_a, t_a) = K(b|a)$, which represents the transition density of price gap going from x_a at time t_a to x_b at time t_b as

$$K(b|a) = \int_{x_a}^{x_b} e^{-\frac{1}{\sigma^2}\int_{t_a}^{t_b} L(\dot{x},x,\tau)d\tau} \mathcal{D}x(\tau),$$

where $L(\dot{x}, x, t) = \frac{1}{2}\dot{x}^2(t) + \sigma^2\Lambda(x,t)$ is the Lagrangian, or equivalently, it is rewritten as

$$K^0(b|a) = \int_{x_a}^{x_b} e^{-\frac{1}{\sigma^2}\int_{t_a}^{t_b}\left[\frac{1}{2}\dot{x}^2(\tau)+\sigma^2\Lambda(x)\right]d\tau} \mathcal{D}x(\tau) \tag{15}$$

with zero inflation and

$$K^{\mu(t)}(b|a) = w_{t_b}(x_b) \int_{x_a}^{x_b} e^{-\frac{1}{\sigma^2}\int_{t_a}^{t_b}\left[\frac{1}{2}\dot{x}^2(\tau)+\sigma^2\Lambda(x,\tau)\right]d\tau} \mathcal{D}x(\tau) \tag{16}$$

in the presence of time-varying inflation. Here, $x(t)$ denotes the any possible transitional path of price gap x from time t_a to time t_b. $\mathcal{D}$ explicitly refers to the fact that the integral is taken with respect to all the possible paths of x between x_a and x_b.

One of the biggest shortcomings of Brownian motion (Weiner) processes based on which KFE is formulated is that the Brownian stochastic process is not differentiable with respect to time. We will show that any Brownian stochastic process can be equivalently formulated by path integrals by an equivalence of KFE and path integral formulation in this regard. Here, we aim to show that any path integral formulated stochastic process which turns out to be an equivalence of Brownian stochastic process turns out to be differentiable with respect to time. Indeed, an intriguing analytical feature of the path integral formulation is that it makes $x(t)$ differentiable everywhere with respect to time t, i.e., it makes $\dot{x}(t)$ a real continuous function of time t, from x_a at time t_a to x_b at time t_b by rewriting $x(t) = \bar{x}(t) + y(t)$ even with Brownian $dx(t)$, because the deterministic least-action path $\bar{x}(t)$ determined by EL equation is differentiable everywhere with respect to t from x_a to x_b. Meanwhile, the perturbed $y(t)$ with $y(t_a) = y(t_b) = 0$, without loss of generality, can be expressed in terms of Fourier series as

$$y(t) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi(t-t_a)}{t_b-t_a}\right) \tag{17}$$

with which coefficients a_n are random coefficients. Note that $y(t)$ written in the form of Fourier series above is a continuously differentiable function of t from x_a at time t_a to x_b at time t_b. As a result,

$$\begin{aligned} \dot{x}(t) &= \dot{\bar{x}}(t) + \dot{y}(t) \\ &= \dot{\bar{x}}(t) + \frac{\pi}{t_b-t_a} \sum_{n=1}^{\infty} n a_n \cos\left(\frac{n\pi(t-t_a)}{t_b-t_a}\right), \end{aligned} \tag{18}$$

which is obviously not only a continuous function of t but also differentiable with respect to t everywhere from x_a at time t_a to x_b at time t_b. From the perspective of making $x(t)$ differentiable with respect to t everywhere from x_a at time t_a to x_b at time t_b through path integral formulation by writing $x(t) = \bar{x}(t) + y(t)$ with $y(t_a) = y(t_b) = 0$ even with Brownian $dx(t)$, we break a ground for any further analytical exploration of the time path of $x(t)$ which is usually not differentiable everywhere with respect to t due to the Brownian process followed by $dx(t)$.

Since for both cases of zero inflation and time-varying inflation, the transition density of price gap $K(b|a)$ can be written in terms of $x(t) = \bar{x}(t) + y(t)$, the transition density formulations given above can thus be rewritten in terms of the least-action deterministic path $\bar{x}(t)$ from a to b and the perturbed path $y(t)$ from a to b. Also note that the path integrals treat the least-action deterministic path $\bar{x}(t)$ as a reference path or a constant path relative to the perturbed path $y(t)$ where fixing $y_a = y_b = 0$, it follows that the transition density $K(b|a) = K(x_b, t_b; x_a, t_a)$ above can be eventually written in the case of zero inflation with an implied state-dependent generalized hazard function $\Lambda(x)$ as

$$\begin{aligned} K^0(b|a) &= \int_{x_a}^{x_b} e^{-\frac{1}{\sigma^2}\int_{t_a}^{t_b}\left[\frac{1}{2}\dot{x}^2(\tau)+\sigma^2\Lambda(x)\right]d\tau}\mathcal{D}x(\tau) \\ &= e^{-\frac{1}{\sigma^2}S^0[\bar{x}(t)]}\int_0^0 e^{-\frac{1}{\sigma^2}S^0[y(t)]}\mathcal{D}y(t) \end{aligned} \tag{19}$$

and written in the presence of time-varying inflation with an implied state- and time-dependent generalized hazard function $\Lambda(x,t)$ as

$$\begin{aligned} K^{\mu(t)}(b|a) &= e^{\frac{\mu(t_b)}{\sigma^2}x_b}\int_{x_a}^{x_b} e^{-\frac{1}{\sigma^2}\int_{t_a}^{t_b}\left[\frac{1}{2}\dot{x}^2(\tau)+\sigma^2\Lambda(x,\tau)\right]d\tau}\mathcal{D}x(\tau) \\ &= e^{-\frac{1}{2\sigma^2}\int_{t_a}^{t_b}\mu^2(t)dt}e^{\frac{\mu(t_b)}{\sigma^2}x_b}e^{-\frac{1}{4\kappa}\int_{t_a}^{t_b}f^2(t)dt} \\ &\times e^{-\frac{1}{\sigma^2}S^{\mu(t)}[\bar{x}(t)]}\int_0^0 e^{-\frac{1}{\sigma^2}S^{\mu(t)}[y(t)]}\mathcal{D}y(t), \end{aligned} \tag{20}$$

where both the integrals above are taken with respect to the perturbed path $y(t)$ denoted by $\mathcal{D}y(t)$ rather than with respect to the ordinary integral usually denoted by dy in which y is not a stochastic path but an interval of real numbers. Hence, in the formulation of path integrals, we see the path integrals taken with respect to a stochastic (or perturbed) path $y(t)$ as $e^{-\frac{1}{\sigma^2}S[\bar{x}(t)]}\int_0^0 e^{-\frac{1}{\sigma^2}S[y(t)]}\mathcal{D}y(t)$ generally is not zero, because path integrals restricting two end points at zero just means the two end points, i.e., y_a and y_b, of the stochastic path $y(t)$ are fixed relative to the least-action deterministic reference path $\bar{x}(t)$. However, if all this is done in the sense of the ordinary integrals taken with respect to a real-valued interval in which y is any real number rather than a stochastic path $y(t)$, i.e., $\int_0^0 e^{-\frac{1}{\sigma^2}S[y]}dy$, then the result is always zero. In the proof for Proposition 1 in Appendix, we show how exactly the path integrals are performed with respect to a perturbed path $y(t)$ when fixing two end points of the perturbed path $y(t)$, y_a and y_b, relative to the least-action deterministic reference path $\bar{x}(t)$, i.e., $e^{-\frac{1}{\sigma^2}S[\bar{x}(t)]}\int_0^0 e^{-\frac{1}{\sigma^2}S[y(t)]}\mathcal{D}y(t)$.

We then can simply utilize the path integral formulation outlined above to analytically derive the transition density of the price gap in the context of sticky prices with quadratic generalized hazard function not only in the case of zero inflation but also in the presence of time-varying inflation. Next proposition gives the analytical transition density of price gap in the context of sticky prices with quadratic generalized hazard function in the case of zero inflation.

Proposition 1. *The path integral transition density of state variable of price gap x, $K^0(x_b, t_b; x_a, t_a)$, or simply $K^0(b|a)$, going from x_a at time t_a to x_b at time t_b following a monetary shock that occurs at time $t = t_a$ to the sticky-price economy with an implied quadratic generalized hazard function $\Lambda(x) = \kappa x^2$ and volatility σ is given by*

$$K^0(b|a) = \left[\frac{\sqrt{2\kappa}}{2\pi\sigma \sinh \sqrt{2\kappa}\sigma(t_b - t_a)}\right]^{1/2} e^{-\frac{\sqrt{2\kappa}}{2\sigma}\left[\frac{\left(x_a^2+x_b^2\right)\cosh\sigma\sqrt{2\kappa}(t_b-t_a)-2x_a x_b}{\sinh\sigma\sqrt{2\kappa}(t_b-t_a)}\right]}. \tag{21}$$

Proof. See Appendix. □

From the path integral transition density of price gap in the case of zero inflation, $K^0(b|a)$, as in Proposition 1, we easily see that $K^0(b|a) = K^0(a|b)$, meaning the transition density of sticky-price gap in the case of zero inflation is symmetric.

3 Path integrals, KFE, and eigenvalue-eigenfunction decomposition

This section lays out the theoretical foundation of path integral formulation in its relation to the KFE formulation and spectral (eigenvalue-eigenfunction) decomposition. Specifically, we first examine the path integral formulation in its relation to the KFE formulation and then explore path integral formulation in its relation to spectral (eigenvalue-eigenfunction) decomposition.

3.1 Path integral formulation and KFE formulation

This subsection simply shows that path integral formulation is a more general and advanced technique compared to the KFE formulation in the sense that path integral formulation of sticky prices with any generalized hazard function (i.e., not just quadratic generalized hazard function) implies the corresponding KFE formulation, while the KFE formulation does not imply the path integral formulated solutions.

Proposition 2. *Path integral formulation of sticky prices with any state- and time-dependent generalized hazard function $\Lambda(x,t)$ implies the corresponding KFE formulation which further leads to the (spectral) eigenvalue-eigenfunction decomposition, but not vice versa.*

Proof. Note that for a short time interval ϵ, by path integral formulation (see section 4 of this paper and equation 2.34 on page 38 in Feynman and Hibbs (1965)), time-$(t+\epsilon)$ density of price gap $p(x,t+\epsilon)$ can be written in terms of time-t density of price gap $p(y,t)$ as

$$p(x,t+\epsilon)=\frac{1}{A}\int_{-\infty}^{\infty}e^{-\frac{1}{\sigma^2}\epsilon L\left(\frac{x-y}{\epsilon},\frac{x+y}{2}\right)}p(y,t)dy, \tag{22}$$

where $L(\dot{x},x)=\frac{1}{2}\dot{x}^2+\sigma^2\Lambda(x,t)$, so that $L\left(\frac{x-y}{\epsilon},\frac{x+y}{2}\right)=\frac{1}{2}\left(\frac{x-y}{\epsilon}\right)^2+\sigma^2\Lambda\left(\frac{x+y}{2},t\right)$. And thus by plugging in, equation above can be rewritten as

$$p(x,t+\epsilon)=\frac{1}{A}\int_{-\infty}^{\infty}e^{-\frac{1}{\sigma^2}\frac{(x-y)^2}{2\epsilon}}\times e^{-\frac{1}{\sigma^2}\epsilon\sigma^2\Lambda\left(\frac{x+y}{2},t\right)}p(y,t)dy \tag{23}$$

By a change of variables $y=x+\eta$, we have

$$p(x,t+\epsilon)=\frac{1}{A}\int_{-\infty}^{\infty}e^{-\frac{\eta^2}{2\sigma^2\epsilon}}\times e^{-\frac{1}{\sigma^2}\epsilon\sigma^2\Lambda\left(x+\frac{\eta}{2},t\right)}p(x+\eta,t)d\eta \tag{24}$$

By expanding ψ in a power series, we get

$$p(x,t)+\epsilon\frac{\partial p}{\partial t}=\frac{1}{A}\int_{-\infty}^{\infty}e^{-\frac{\eta^2}{2\sigma^2\epsilon}}\times\left[1-\frac{1}{\sigma^2}\epsilon\sigma^2\Lambda(x,t)\right]\left[p(x,t)+\eta\frac{\partial p}{\partial x}+\frac{\eta^2}{2}\frac{\partial^2 p}{\partial x^2}\right]d\eta \tag{25}$$

Note that the leading term on the right-hand side is equal to (by Gaussian integral)

$$\frac{1}{A}\int_{-\infty}^{\infty}e^{-\frac{\eta^2}{2\sigma^2\epsilon}}d\eta=\frac{1}{A}\left(2\pi\sigma^2\epsilon\right)^{1/2} \tag{26}$$

On the left-hand side, there is only $p(x,t)$, therefore, to let both sides agree to each other, A must be chosen so that $\frac{1}{A}(2\pi\sigma^2\epsilon)^{1/2}=1$, that is,

$$A=\left(2\pi\sigma^2\epsilon\right)^{1/2} \tag{27}$$

Moreover, we can calculate the other two terms on the right-hand side of the expanded equation, that is,

$$\frac{1}{A}\int_{-\infty}^{\infty}\eta e^{-\frac{\eta^2}{2\sigma^2\epsilon}}d\eta = 0 \tag{28}$$

$$\frac{1}{A}\int_{-\infty}^{\infty}\eta^2 e^{-\frac{\eta^2}{2\sigma^2\epsilon}}d\eta = \sigma^2\epsilon \tag{29}$$

Finally, writing out the full version of the expanded equation using the fact that the second order of ϵ goes to zero, that is, $\epsilon^2 \to 0$, we get

$$p(x,t) + \epsilon\frac{\partial p}{\partial t} = p(x,t) - \frac{1}{\sigma^2}\epsilon\sigma^2\Lambda(x,t)p(x,t) + \frac{\sigma^2\epsilon}{2}\frac{\partial^2 p}{\partial x^2} \tag{30}$$

Simplifying it, we get

$$\frac{\partial p}{\partial t} = \frac{\sigma^2}{2}\frac{\partial^2 p}{\partial x^2} - \Lambda(x,t)p(x,t), \tag{31}$$

which is the corresponding KFE equation characterizing time evolution of density of the price gap with zero inflation and with any form of both state- and time-dependent generalized hazard function $\Lambda(x,t)$. In particular, if the generalized hazard function is quadratic, i.e., $\Lambda(x) = \kappa x^2$, it is widely known that (see Alvarez and Lippi (2022)) the eigenvalue-eigenfunction decomposed solution to the KFE (31), i.e., eigenvalues λ_j and eigenfunctions $\phi_j(x)$ of the transition matrix $\mathcal{B}$, are given by

$$\lambda_j = \sigma\sqrt{2\kappa}\left(j - \frac{1}{2}\right), \tag{32}$$

and

$$\phi_j(x) = \frac{1}{\pi^{1/4}(2^{j-1}(j-1)!)^{1/2}}\left(\frac{2\kappa}{\sigma^2}\right)^{1/8} H_{j-1}\left(\left(\frac{2\kappa}{\sigma^2}\right)^{1/4}x\right)e^{-\left(\frac{\kappa}{2\sigma^2}\right)^{1/2}x^2}, \tag{33}$$

where $j = 1,2,3,...$ and $H_{j-1}(\cdot)$ is the Hermite polynomial of degree $j-1$.

Hence, we have proven that path integral formulation implies KFE formulation which may further lead to a spectral (eigenvalue-eigenfunction) decomposed solution which will be shown in Proposition 3. Moreover, in Proposition 4, we will show that the spectral decomposition (i.e., the eigenvalues λ_j and eigenfunctions $\phi_j(x)$) obtained in Alvarez and Lippi (2022) with quadratic generalized hazard function as mentioned above can be directly derived from the path integral transition density $K^0(y|x)$ as expressed in Proposition 1 without having to resort to the KFE approach as taken by Alvarez and Lippi (2022). □

3.2 Path integral formulation and eigenvalue-eigenfunction decomposition

We next prove path integral formulation of the sticky-price economy with generalized hazard function $\Lambda(x)$ and volatility of the economy σ implies the corresponding spectral (eigenvalue-eigenfunction) decomposition of the transition matrix $\mathcal{B}$ of the economy. Or specifically, it is given by the following proposition as

Proposition 3. *The path integral transition density of price gap following a monetary shock to a sticky-price economy of zero inflation with generalized hazard function $\Lambda(x)$ and volatility σ, $K^0(y|x)$, from x at time 0 to y at time t, can be equivalently written in terms of the (spectral) eigenvalue-eigenfunction decomposed transition density, $\sum_{j=1}^{\infty} e^{-\lambda_j t}\phi_j(x)\phi_j(y)$, as*

$$K^0(y|x) = \sum_{j=1}^{\infty} e^{-\lambda_j t}\phi_j(x)\phi_j(y). \tag{34}$$

Proof. Since by definition our eigenfunctions $\phi_j(x)$ and $\phi_i(x)$, where $i, j = 1, 2, 3, ...$, are orthonormal, it follows that we have

$$\int_{-\infty}^{\infty} \phi_j(x)\phi_i(x)dx = \delta_{j,i},$$

where $\delta_{j,i}$, the Kronecker delta, is defined by $\delta_{j,i} = 0$ if $j \neq i$ and $\delta_{j,i} = 1$ if $j = i$. Therefore, any function $f(x)$ can be expressed in terms of the eigenfunctions $\phi_j(x)$ which form an orthonormal basis, that is,

$$f(x) = \sum_{j=1}^{\infty} a_j\phi_j(x).$$

The coefficients a_j are thus obtained by

$$a_j = \int_{-\infty}^{\infty} f(x)\phi_j(x)dx.$$

Thus, by plugging in a_j, $f(x)$ is rewritten as

$$f(x) = \sum_{j=1}^{\infty} \phi_j(x) \int_{-\infty}^{\infty} f(y)\phi_j(y)dy = \int_{-\infty}^{\infty} \left[\sum_{j=1}^{\infty} \phi_j(x)\phi_j(y) \right] f(y)dy.$$

Now, consider the solution $p(x,t)$ taking the form of linear combination of $e^{-\lambda_j t}\phi_j(x)$, and hence we have

$$p(x,t) = \sum_{j=1}^{\infty} c_j e^{-\lambda_j t}\phi_j(x).$$

Particularly, at time $t = t_a$, by letting $f(x) = p(x,t_a)$, we have

$$f(x) = p(x,t_a) = \sum_{j=1}^{\infty} c_j e^{-\lambda_j t_a}\phi_j(x) = \sum_{j=1}^{\infty} a_j \phi_j(x).$$

Therefore, we get

$$c_j = a_j e^{\lambda_j t_a}.$$

Note that at time $t = t_b > t_a$ the solution $p(x,t_b)$ is analogously written as

$$p(x,t_b) = \sum_{j=1}^{\infty} c_j e^{-\lambda_j t_b}\phi_j(x).$$

Hence, by plugging in c_j, we get

$$p(x,t_b) = \sum_{j=1}^{\infty} a_j e^{-\lambda_j (t_b - t_a)}\phi_j(x).$$

Also note that

$$a_j = \int_{-\infty}^{\infty} \phi_j(y)f(y)dy,$$

by plugging in, we get

$$\begin{aligned} p(x,t_b) &= \sum_{j=1}^{\infty} \phi_j(x)e^{-\lambda_j (t_b - t_a)} \int_{-\infty}^{\infty} \phi_j(y)f(y)dy \\ &= \int_{-\infty}^{\infty} \sum_{j=1}^{\infty} \phi_j(x)\phi_j(y)e^{-\lambda_j (t_b - t_a)} f(y)dy. \end{aligned}$$

Finally, by noting that the solution $p(x,t_b)$ can also be written in terms of the transition density $K^0(x|y)$ with which the transition is from y at time t_a to x at time

t_b with $(t_b > t_a)$, that is,

$$p(x, t_b) = \int_{-\infty}^{\infty} K^0(x|y) f(y) dy.$$

Comparing the last two equations regarding $p(x, t_b)$, we get

$$K^0(x|y) = \sum_{j=1}^{\infty} \phi_j(x)\phi_j(y)e^{-\lambda_j(t_b - t_a)}.$$

Note that $K^0(y|x) = K^0(x|y)$ by symmetry implied by Proposition 1, we eventually get our desired result in a more general version as

$$K^0(y|x) = \sum_{j=1}^{\infty} \phi_j(x)\phi_j(y)e^{-\lambda_j(t_b - t_a)}.$$

If we let $t_a = 0$ and $t_b = t$, then we get

$$K^0(y|x) = \sum_{j=1}^{\infty} e^{-\lambda_j t}\phi_j(x)\phi_j(y).$$

□

Proposition 3 provides an equivalence of the path integral transition density and the (spectral) eigenvalue-eigenfunction decomposed transition density in the case of zero inflation of the sticky-price economy. Next, we give Proposition 4 which works on a special case of Proposition 3 with quadratic generalized hazard function $\Lambda(x) = \kappa x^2$. The significance of Proposition 4 relies on its usefulness that allows us to directly derive the spectral (eigenvalue-eigenfunction) decomposition of the transition matrix $\mathcal{B}$ in the case of quadratic generalized hazard function from the path integral transition density $K^0(y|x)$ without having to resort to the KFE formulation to find out $\mathcal{B}\phi = \lambda\phi$ where λ and ϕ are the respective eigenvalue and eigenfunction as in the case of Alvarez and Lippi (2022).

Proposition 4. *Following Proposition 3, as a special case with quadratic generalized hazard function $\Lambda(x) = \kappa x^2$, the path integral transition density $K^0(y|x)$ as obtained in Proposition 1 implies the following spectral (eigenvalue-eigenfunction) decomposi-*

tion

$$
\begin{aligned}
K^0(y|x) &= \left[\frac{\sqrt{2\kappa}}{2\pi\sigma\sinh\sqrt{2\kappa}\sigma t}\right]^{1/2} e^{-\frac{\sqrt{2\kappa}}{2\sigma}\left[\frac{\left(x^2+y^2\right)\cosh\sigma\sqrt{2\kappa}t-2xy}{\sinh\sigma\sqrt{2\kappa}t}\right]} \\
&= \sum_{j=1}^{\infty} e^{-\sigma\sqrt{2\kappa}\left(j-\frac{1}{2}\right)t} \\
&\times \frac{1}{\pi^{1/4}(2^{j-1}(j-1)!)^{1/2}}\left(\frac{2\kappa}{\sigma^2}\right)^{1/8} H_{j-1}\left(\left(\frac{2\kappa}{\sigma^2}\right)^{1/4} x\right) e^{-\left(\frac{\kappa}{2\sigma^2}\right)^{1/2}x^2} \\
&\times \frac{1}{\pi^{1/4}(2^{j-1}(j-1)!)^{1/2}}\left(\frac{2\kappa}{\sigma^2}\right)^{1/8} H_{j-1}\left(\left(\frac{2\kappa}{\sigma^2}\right)^{1/4} y\right) e^{-\left(\frac{\kappa}{2\sigma^2}\right)^{1/2}y^2},
\end{aligned}
\tag{35}
$$

where $j = 1, 2, 3, ...,$ *and* $H_{j-1}(\cdot)$ *is the Hermite polynomial of degree* $j-1$*, and the eigenvalues are* $\lambda_j = \sigma\sqrt{2\kappa}(j-\frac{1}{2})$ *and the eigenfunctions are* $\phi_j(x) = \frac{1}{\pi^{1/4}(2^{j-1}(j-1)!)^{1/2}}\left(\frac{2\kappa}{\sigma^2}\right)^{1/8} H_{j-1}\left(\left(\frac{2\kappa}{\sigma^2}\right)^{1/4} x\right) e^{-\left(\frac{\kappa}{2\sigma^2}\right)^{1/2}x^2}$.

Proof. See Appendix. □

From Proposition 4, we can clearly see that the transition density of price gap in the case of zero inflation with quadratic generalized hazard function, $K^0(y|x)$, is symmetric. That is, $K^0(y|x) = K^0(x|y)$. We can directly see this by switching x and y in the path integral formulated transition density which does not take eigenvalue-eigenfunction decomposition form. A better way of examining this is by switching x and y in the (spectral) eigenvalue-eigenfunction decomposed transition density on the right hand side of the equation (35). The symmetry of the transition density of price gap in the case of zero inflation is an intriguing property that any monetary authority should pursue by effectively implementing its monetary tools, because the symmetry automatically triggers a symmetric steady-state distribution of the state variable of price gap following a monetary shock, which makes the whole system of the economy during the transition only have one steady state rather than multiple steady states. In other words, if the transition density of price gap is asymmetric, then following the monetary shock there would exist an asymmetric steady-state distribution of price gap which is different from the initial symmetric steady-state distribution of the price gap.

Transition density and the reinjections of firms with zero inflation. We aim to show that the transition density of price gap, in the presence of zero inflation, keeps the same version regardless of whether the reinjections of the firms are considered or not (i.e., regardless of keeping track of firms until the stopping time τ at which the firm resets price or tracking all firms even after the stopping time τ). That is, the transition density of price gap without firm's reinjection $K^0(y|x)$ is equivalent to the transition density of price gap with firm's reinjection $\mathcal{K}^0(y|x)$ if the inflation is zero, i.e., $K^0(y|x) = \mathcal{K}^0(y|x)$ with $\mu(t) = 0$ for all t. Here, we have $K^0(y|x)$ and $\mathcal{K}^0(y|x)$ solving the KFE

$$\partial_t K^0(y|x) = (\sigma^2/2)\partial_y^2 K^0(y|x) - \Lambda(y)K^0(y|x). \tag{36}$$

and

$$\partial_t \mathcal{K}^0(y|x) = (\sigma^2/2)\partial_y^2 \mathcal{K}^0(y|x) - \Lambda(y)\mathcal{K}^0(y|x) + \Lambda(y)\delta_0(y), \tag{37}$$

respectively.

To prove $K^0(y|x) = \mathcal{K}^0(y|x)$, we assume that the eigenfunctions of the transition matrix $\mathcal{B}$ are $\phi_j(x)$ and the eigenvalues of the transition matrix $\mathcal{B}$ are λ_j, where $j = 1, 2, 3, ...$ Then it follows that the transition density of price gap $K^0(y|x)$ without firm's reinjection, is written as

$$K^0(y|x) = \sum_{j=1}^{\infty} e^{-\lambda_j t}\phi_j(x)\phi_j(y), \tag{38}$$

and the transition density $\mathcal{K}^0(y|x)$, i.e., the transition density of price gap with firm's reinjection, is written as

$$\mathcal{K}^0(y|x) = \sum_{j=1}^{\infty} a_j(t)\phi_j(x)\phi_j(y) \tag{39}$$

Now, if we can show that $a_j(t) = e^{-\lambda_j t}$, then we are done. Next, we show they are indeed equal. Note that

$$a_j(t) = a_j(0)e^{-\lambda_j t} + \int_0^t q_j(\tau)e^{\lambda_j(\tau - t)}d\tau, \tag{40}$$

where

$$q_j(t) = \int_{-\infty}^{\infty} \Lambda(x)\delta_0(y)\phi_j(x)dx. \tag{41}$$

Since $\int_{-\infty}^{\infty} \Lambda(x)\delta_0(y)\phi_j(x)dx = \Lambda(0)\phi_j(0) = 0$ given $\Lambda(0) = 0$, it follows that $q_j(t) = 0$, and therefore, $a_j(t) = a_j(0)e^{-\lambda_j t}$. Now consider $a_j(0) = \int_{-\infty}^{\infty} \mathcal{K}_0^0(x|x)\phi_j(x)dx$, where $\mathcal{K}_0^0(x|x)$ denotes the initial condition of transition density, $\mathcal{K}^0(y|x)$, which, by definition, is equal to $\phi_j(x)$, that is, $\mathcal{K}_0^0(x|x) = \phi_j(x)$. Hence, we have $a_j(0) = \int_{-\infty}^{\infty} \mathcal{K}_0^0(x|x)\phi_j(x)dx = \int_{-\infty}^{\infty} \phi_j^2(x)dx = 1$ because the eigenfunctions $\phi_j(x)$ form an orthonormal basis. Therefore, we get $a_j(t) = e^{-\lambda_j t}$, and hence $K^0(y|x) = \mathcal{K}^0(y|x)$.

4 A monetary shock and path integral formulation

We study the analytical marginal impulse response of output following a monetary shock in the case of zero inflation. Since in this section we consider the quadratic hazard function $\Lambda(x) = \kappa x^2$, we must first solve for the initial steady state of the distribution of the price gap. Simply note that the time-independent KFE characterizing the stationary distribution of the price gap with a quadratic hazard function is written as $\kappa x^2 f(x) = \frac{\sigma^2}{2} f''(x)$, where $f(x)$ is the stationary distribution of price gap x and σ is a measure of cost uncertainty of the economy. The solution to this time-independent KFE takes the following form

$$\begin{aligned} f(x) &= e^{-\sqrt{\kappa/2\sigma^2}x^2} \sum_{j=0}^{\infty} a_{2j} x^{2j} \\ &= a_0 e^{-\sqrt{\kappa/2\sigma^2}x^2} + e^{-\sqrt{\kappa/2\sigma^2}x^2} \sum_{j=1}^{\infty} a_{2j} x^{2j} \end{aligned} \tag{42}$$

where a_0 is a normalization factor and is determined by the normalization of density $f(x)$ (i.e., $\int_{-\infty}^{\infty} f(x)dx = 1$) and the coefficients a_{2j} are recursively given by

$$a_{2j+2} = \frac{4\sqrt{\kappa/2\sigma^2}j + \sqrt{\kappa/2\sigma^2}}{(2j+1)(j+1)} a_{2j}, \tag{43}$$

note that all coefficients a_{2j} will be completely determined by normalization factor a_0. We thus can write the output marginal impulse response function as

$$\mathcal{Y}(t) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (-y) K^0(y|x) f'(x) dx dy,$$

where $K^0(y|x)$ is the transition density of price gap following a monetary shock in the case of zero inflation, that is

$$K^0(y|x) = \left[\frac{\sqrt{2\kappa}}{2\pi\sigma \sinh\sqrt{2\kappa}\sigma t}\right]^{1/2} e^{-\frac{\sqrt{2\kappa}}{2\sigma}\left[\frac{\left(x^2+y^2\right)\cosh\sigma\sqrt{2\kappa}t-2xy}{\sinh\sigma\sqrt{2\kappa}t}\right]},$$

and $f'(x)$ is the derivative of the invariant density of price gap which is given by

$$f'(x) = a_0(-2)\sqrt{\kappa/2\sigma^2}xe^{-\sqrt{\kappa/2\sigma^2}x^2} + (-2)\sqrt{\kappa/2\sigma^2}xe^{-\sqrt{\kappa/2\sigma^2}x^2}\sum_{j=1}^{\infty} a_{2j}x^{2j}$$
$$+ e^{-\sqrt{\kappa/2\sigma^2}x^2}\sum_{j=1}^{\infty}(2j)a_{2j}x^{2j-1}.$$

4.1 Asymptotic hazard rate and the dominant eigenvalue λ_1

We aim to provide an equivalence of the asymptotic hazard rate and the dominant eigenvalue in the context of $\mu(t) = 0$ with quadratic generalized hazard function. But before doing that, we need to start from the duration-based survival function based on which the asymptotic hazard rate is derived. As discussed in the Appendix of Alvarez, Lippi, and Oskolkov (2022), "duration-based functions are often used in sticky price models and it is interesting to know whether the information encoded in them is different from that encoded in the size distribution pf price gaps". Empirically speaking, the survival function defined as $S(t) = Pr(t \leq \tau)$, where τ is the first stopping time at which the firm resets its prices, that is used to characterize the probability that a price spell lasts at least t units of time is either observable or measurable in data.

Alvarez, Lippi, and Oskolkov (2022) study the survival function with quadratic hazard function in a zero inflation setting using Feynman-Kac formula and Alvarez and Lippi (2022) study their version of survival function in a zero inflation setting using eigenvalue-eigenfunction decomposition approach. This paper introduces path integrals to study the survival function $S(t)$. This application deals with the survival function $S(t)$ with quadratic generalized hazard function $\Lambda(x) = \kappa x^2$ in a zero inflation setting using the transition density obtained above, $K(x_b, t_b; x_a, t_a)$. Recall the transition density of price gap x from x_a to x_b during periods $t_b - t_a$ with zero inflation is given by

$$K(x_b, t_b; x_a, t_a) = \left[\frac{\sqrt{2\kappa}}{2\pi\sigma \sinh \sqrt{2\kappa}\sigma(t_b - t_a)}\right]^{1/2} e^{-\frac{\sqrt{2\kappa}}{2\sigma}\left[\frac{\left(x_a^2+x_b^2\right)\cosh\sigma\sqrt{2\kappa}(t_b-t_a)-2x_a x_b}{\sinh\sigma\sqrt{2\kappa}(t_b-t_a)}\right]}.$$

We assume the first stopping time τ starts at time zero, and thus the survival function $S(t)$ is obtained as

$$\begin{aligned} S(t) = Pr(t \leq \tau) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} K(x, t|y, 0)\delta_0(y)dxdy \\ &= \int_{-\infty}^{\infty} K(x, t|0, 0)dx \\ &= \int_{-\infty}^{\infty}\left[\frac{\sqrt{2\kappa}}{2\pi\sigma \sinh \sqrt{2\kappa}\sigma t}\right]^{1/2} e^{-\frac{\sqrt{2\kappa}}{2\sigma}\left(\frac{\cosh\sigma\sqrt{2\kappa}t}{\sinh\sigma\sqrt{2\kappa}t}\right)x^2} dx \\ &= \sqrt{\frac{1}{\cosh\sigma\sqrt{2\kappa}t}}, \end{aligned} \tag{44}$$

where $\delta_0(y)$ is the Dirac delta function centered at 0 and the notation $K(x, t|y, 0)$ is equivalent to $K(x, t; y, 0)$.

Given the survival function, we can directly calculate the hazard rate of price changes defined as $h(t) = -S'(t)/S(t)$ as

$$h(t) = -\frac{S'(t)}{S(t)} = \frac{\sigma\sqrt{2\kappa}}{2}\tanh\sigma\sqrt{2\kappa}t \tag{45}$$

and therefore the asymptotic hazard rate $\lim_{t\to\infty} h(t)$ is given by

$$\lim_{t\to\infty} h(t) = \frac{\sigma\sqrt{2\kappa}}{2}\lim_{t\to\infty}\tanh\sigma\sqrt{2\kappa}t = \frac{1}{2}\sigma\sqrt{2\kappa} = \lambda_1, \tag{46}$$

where we have used the fact from Proposition 4 that the dominant eigenvalue of the transition dynamics of the sticky-price economy with quadratic generalized hazard function $\Lambda(x) = \kappa x^2$ in the case of $\mu(t) = 0$ is λ_1 given by $\lambda_1 = \frac{1}{2}\sigma\sqrt{2\kappa}$. Here, we have shown that the asymptotic hazard rate is equal to the dominant eigenvalue of the corresponding transition dynamics of the economy, which is one of the analytical results from Corollary 3 in Alvarez and Lippi (2022). The novelty here is that we use the path integral formulated transition density $K(b|a)$ to come up with the same analytical result as in Alvarez and Lippi (2022) where they use the eigenvalue-eigenfunction decomposition to figure out the result.

4.2 The average speed of convergence and the leading eigenvalue λ_2

This section studies the average speed of convergence of the transition dynamics of the output in a sticky-price economy with quadratic generalized hazard function $\Lambda(x) = \kappa x^2$ in the case of zero inflation following a monetary shock. Given the marginal output impulse response function following a monetary shock in the aforementioned context

$$\mathcal{Y}^0(t) = \sum_{j=1}^{\infty} e^{-\lambda_j t} \left[\int_{-\infty}^{\infty} (-y)\phi_j(y)dy \right] \left[\int_{-\infty}^{\infty} f'(x)\phi_j(x)dx \right],$$

where

$$\lambda_j = \sigma\sqrt{2\kappa}\left(j - \frac{1}{2}\right),$$

$$\phi_j(x) = \frac{1}{\pi^{1/4}(2^{j-1}(j-1)!)^{1/2}} \left(\frac{2\kappa}{\sigma^2}\right)^{1/8} H_{j-1}\left(\left(\frac{2\kappa}{\sigma^2}\right)^{1/4} x\right) e^{-\left(\frac{\kappa}{2\sigma^2}\right)^{1/2} x^2},$$

and $f'(x)$ is first-order derivative of the stationary density of price gap, the average speed of convergence of this marginal output impulse response $\lambda^0(y)$ is expressed as

$$\lambda^0(y) = -\lim_{t\to\infty} \frac{1}{t} \log\big(\mathcal{Y}^0(t) - \mathcal{Y}^0_\infty\big), \tag{47}$$

where $\mathcal{Y}^0_\infty$ is the marginal output impulse response when $t \to \infty$ and we know that in the case of zero inflation, $\mathcal{Y}^0_\infty = 0$ (i.e., the new steady state following a monetary shock is the same as the original steady state of the economy). Hence, the average speed of convergence in this case can be written as

$$\lambda^0(y) = -\lim_{t\to\infty} \frac{1}{t} \log\left\{ \sum_{j=1}^{\infty} e^{-\lambda_j t} \left[\int_{-\infty}^{\infty} (-y)\phi_j(y)dy \right] \left[\int_{-\infty}^{\infty} f'(x)\phi_j(x)dx \right] \right\}. \tag{48}$$

Here comes a very important observation that has also been emphasized in Alvarez and Lippi (2022) right after Corollary 3, which is that $\int_{-\infty}^{\infty}(-y)\phi_j(y)dy = 0$ for all odd $j = 1, 3, 5, ...$, because for any odd j, $\phi_j(x)$ is an even function due to the fact that $H_{(j-1)=n=even}(\cdot)$ is an even function for j odd and thus $\int_{-\infty}^{\infty}(-y)\phi_j(y)dy = 0$ for

j odd. Therefore, the only non-zero terms in the marginal output impulse response function $\mathcal{Y}^0(t)$ are the even-indexed terms (i.e., $j = 2, 4, 6, ..., .$). Finally, we can evaluate equation (48) to obtain the average speed of convergence in this case as

$$\lambda^0(y) = \lambda_2 = \frac{3}{2}\sigma\sqrt{2\kappa}. \tag{49}$$

This result is also consistent with Alvarez and Lippi (2022) in that they call λ_2 the leading eigenvalue which reflects the inverse of half life of the transition dynamics of output, or equivalently, we call it the average speed of convergence of transition dynamics of output. The novelty here also relies on the fact that the path integrals technique can be used by this paper to derive the exactly same analytical result as in Alvarez and Lippi (2022) regarding the average speed of convergence. The two analytical results regarding the dominant and leading eigenvalue can thus be summarized as follows.

Proposition 5. *In the case of sticky prices with zero inflation and an implied state-dependent quadratic generalized hazard function $\Lambda(x) = \kappa x^2$ with cost volatility σ, the asymptotic hazard rate, $\lim_{t\to\infty} h(t)$, is given by the dominant eigenvalue of the transition dynamics of the sticky-price economy, λ_1, that is,*

$$\lim_{t\to\infty} h(t) = \lambda_1 = \frac{1}{2}\sigma\sqrt{2\kappa}.$$

Moreover, the average speed of the convergence of the transition dynamics of this sticky-price economy reflected by the average speed of the convergence in terms of marginal output impulse response, $\lambda^0(y)$, is given by the corresponding leading eigenvalue of the transition dynamics of the sticky-price economy, λ_2, that is,

$$\lambda^0(y) = \lambda_2 = \frac{3}{2}\sigma\sqrt{2\kappa}.$$

5 Conclusion

This paper introduces path integral formulation into the existing macroeconomics of sticky-price models to analytically study the dynamics of the sticky prices with generalized hazard functions in the presence of zero inflation. We view path integral technique as a very promising mathematical tool not only working for the sticky-price models but also working very well for a wide range of macro models with optimal stopping property that can be characterized by generalized hazard functions. For instance, considering a possible extension of the lumpy investment and the aggregate dynamics of lumpy economies with generalized hazard functions as originally studied in Baley and Blanco (2021), path integral formulation can be directly applied and be seamlessly integrated into the exploration for such a topic.

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A Appendix

.1 Proof of Proposition 1

Proof. Define $L(\dot{x}, x, t) = \frac{1}{2}\dot{x}^2 + \sigma^2\Lambda(x)$. Then,

$$L(\dot{x}, x, t) = \frac{1}{2}\dot{x}^2 + \kappa\sigma^2 x^2$$

and we have, by defining $S[x(t)] = \int_{t_a}^{t_b} L(\dot{x}, x, t)dt$,

$$S[x(t)] = S[\bar{x}(t) + y(t)]$$

that is,

$$\begin{aligned}
I &= S[x(t)] \\
&= \int_{t_a}^{t_b} \left(\frac{1}{2}\big(\dot{\bar{x}}^2 + 2\dot{\bar{x}}\dot{y} + \dot{y}^2\big) + \kappa\sigma^2\big(\bar{x}(t) + y(t)\big)^2 \right) dt \\
&= \int_{t_a}^{t_b} \left(\frac{1}{2}\dot{\bar{x}}^2(t) + \kappa\sigma^2\bar{x}^2(t) \right) dt \\
&+ \int_{t_a}^{t_b} \left(\dot{\bar{x}}(t)\dot{y}(t) + \frac{1}{2}\dot{y}^2(t) + 2\kappa\sigma^2\bar{x}(t)y(t) + \kappa\sigma^2 y^2(t) \right) dt
\end{aligned}$$

Note that

$$\begin{aligned}
S_1 &= \int_{t_a}^{t_b} \big(\dot{\bar{x}}(t)\dot{y}(t) + 2\kappa\sigma^2 \bar{x}(t)y(t)\big)dt \\
&= \int_{t_a}^{t_b} \dot{\bar{x}}(t)dy(t) + 2\kappa\sigma^2 \int_{t_a}^{t_b} \bar{x}(t)y(t)dt \\
&= [\dot{\bar{x}}(t)y(t)]_{t_a}^{t_b} - \int_{t_a}^{t_b} \ddot{\bar{x}}(t)y(t)dt + 2\kappa\sigma^2 \int_{t_a}^{t_b} \bar{x}(t)y(t)dt \\
&= -\int_{t_a}^{t_b} \big(2\kappa\sigma^2\bar{x}(t)\big)y(t)dt + 2\kappa\sigma^2 \int_{t_a}^{t_b} \bar{x}(t)y(t)dt \\
&= 0
\end{aligned}$$

where we have used $y(t_a) = y(t_b) = 0$ and from Euler Lagrange equation for $L(\dot{x}, x, t) = \frac{1}{2}\dot{x}^2 + \kappa\sigma^2 x^2$ to get $\ddot{\bar{x}}(t) = 2\kappa\sigma^2\bar{x}(t)$.

Therefore, we get

$$\begin{aligned}
S[x(t)] &= S[\bar{x}(t) + y(t)] \\
&= \int_{t_a}^{t_b} \left(\frac{1}{2}\dot{\bar{x}}^2(t) + \kappa\sigma^2\bar{x}^2(t)\right)dt + \int_{t_a}^{t_b} \left(\frac{1}{2}\dot{y}^2(t) + \kappa\sigma^2 y^2(t)\right)dt \\
&= S[\bar{x}(t)] + S[y(t)]
\end{aligned}$$

where

$$S[\bar{x}(t)] = \int_{t_a}^{t_b} \left(\frac{1}{2}\dot{\bar{x}}^2(t) + \kappa\sigma^2\bar{x}^2(t)\right)dt$$

$$S[y(t)] = \int_{t_a}^{t_b} \left(\frac{1}{2}\dot{y}^2(t) + \kappa\sigma^2 y^2(t)\right)dt$$

Therefore, we finally get

$$\begin{aligned}
K(b, a) &= \int_a^b \exp\left(-\frac{1}{\sigma^2} S[x(t)]\right)\mathcal{D}x(t) \\
&= \int_0^0 \exp\left(-\frac{1}{\sigma^2} S[\bar{x}(t) + y(t)]\right)\mathcal{D}y(t) \\
&= \int_0^0 \exp\left(-\frac{1}{\sigma^2} S[\bar{x}(t)] - \frac{1}{\sigma^2} S[y(t)]\right)\mathcal{D}y(t) \\
&= \exp\left(-\frac{1}{\sigma^2} S[\bar{x}(t)]\right) \int_0^0 \exp\left(-\frac{1}{\sigma^2} S[y(t)]\right)\mathcal{D}y(t)
\end{aligned}$$

That is, given the generalized hazard function $\Lambda(x) = \kappa x^2$, the corresponding kernel is given by

$$K(b,a) = \exp\left(-\frac{1}{\sigma^2}S[\bar{x}(t)]\right)\int_0^0 \exp\left(-\frac{1}{\sigma^2}S[y(t)]\right)\mathcal{D}y(t) \tag{.1}$$

where

$$S[\bar{x}(t)] = \int_{t_a}^{t_b}\left(\frac{1}{2}\dot{\bar{x}}^2(t) + \kappa\sigma^2\bar{x}^2(t)\right)dt$$

$$S[y(t)] = \int_{t_a}^{t_b}\left(\frac{1}{2}\dot{y}^2(t) + \kappa\sigma^2 y^2(t)\right)dt$$

First, we can compute $\int_0^0 \exp\left(-\frac{1}{\sigma^2}S[y(t)]\right)\mathcal{D}y(t)$ using the Fourier series method, and it turns out

$$\begin{aligned}\int_0^0 \exp\left(-\frac{1}{\sigma^2}S[y(t)]\right)\mathcal{D}y(t) &= \int_0^0 \exp\left(-\frac{1}{\sigma^2}\int_{t_a}^{t_b}\left(\frac{1}{2}\dot{y}^2(t) + \kappa\sigma^2 y^2(t)\right)dt\right)\mathcal{D}y(t) \\ &= \left(\frac{\sqrt{2\kappa}\sigma}{2\pi\sigma^2\sinh\sqrt{2\kappa}\sigma(t_b - t_a)}\right)^{1/2}\end{aligned}$$

To calculate $\int_0^0 \exp\left(-\frac{1}{\sigma^2}\int_{t_a}^{t_b}\left(\frac{1}{2}\dot{y}^2(t) + \kappa\sigma^2 y^2(t)\right)dt\right)\mathcal{D}y(t)$, we first note that the path $y(t)$ has to meet the following requirement: $y(t_a = 0) = y(t_b = T) = 0$, and thus we can write $y(t)$ using Fourier series expansion as

$$y(t) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi t}{T}\right) \tag{.2}$$

Next, by direct plugging in and assuming that the time T is divided into discrete steps of length ϵ, our target of equation can be rewritten as

$$
\begin{aligned}
F(T) &= \int_0^0 \exp\left(-\frac{1}{\sigma^2}\int_{t_a}^{t_b}\left(\frac{1}{2}\dot{y}^2(t)+\kappa\sigma^2 y^2(t)\right)dt\right)\mathcal{D}y(t) \\
&= J\frac{1}{A}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\exp\{-\frac{1}{2\sigma^2}\frac{T}{2}\sum_{n=1}^{N}\left[\left(\frac{n\pi}{T}\right)^2+2\kappa\sigma^2\right]a_n^2\} \\
&\times \frac{da_1}{A}\frac{da_2}{A}\cdots\frac{da_N}{A} \\
&= \frac{J}{A}\prod_{n=1}^{N}\int_{-\infty}^{\infty}\exp\{-\frac{1}{2\sigma^2}\frac{T}{2}\sum_{n=1}^{N}\left[\left(\frac{n\pi}{T}\right)^2+2\kappa\sigma^2\right]a_n^2\}\frac{da_n}{A} \\
&\propto \prod_{n=1}^{N}\left(\frac{n^2\pi^2}{T^2}+2\kappa\sigma^2\right)^{-1/2} \\
&= \prod_{n=1}^{N}\left(\frac{n^2\pi^2}{T^2}\right)^{-1/2}\prod_{n=1}^{N}\left(1+\frac{2\kappa\sigma^2T^2}{n^2\pi^2}\right)^{-1/2} \\
&\propto \left(\frac{\sinh\sqrt{2\kappa}\sigma T}{\sigma\sqrt{2\kappa}T}\right)^{-1/2}
\end{aligned}
\tag{.3}
$$

where we have applied Euler formula to the derivation from the second-to-last line to the last line.

$F(T)$ can be written in the form

$$
F(T) = C\left(\frac{\sinh\sqrt{2\kappa}\sigma T}{\sigma\sqrt{2\kappa}T}\right)^{-1/2} \tag{.4}
$$

We consider the case in which $\sqrt{2\kappa}\sigma = 0$, since we already know from the previous derivations about the equivalence of path integral and KFE formulations that $F(T) = \left(\frac{1}{2\pi\sigma^2 T}\right)^{1/2}$ when $\sqrt{2\kappa}\sigma = 0$, which is just the inverse of the normalizing factor A. On the other hand, we also have (by utilizing L'Hopital's rule),

$$
\left(\frac{1}{2\pi\sigma^2T}\right)^{1/2} = \lim_{\sqrt{2\kappa}\sigma\to 0} F(T) = \lim_{\sqrt{2\kappa}\sigma\to 0} C\left(\frac{\sinh\sqrt{2\kappa}\sigma T}{\sigma\sqrt{2\kappa}T}\right)^{-1/2} = C \tag{.5}
$$

Therefore, our desired integral $F(T)$ is equal to

$$\begin{aligned} F(T) &= \left(\frac{1}{2\pi\sigma^2 T}\right)^{1/2}\left(\frac{\sinh\sqrt{2\kappa}\sigma T}{\sigma\sqrt{2\kappa}T}\right)^{-1/2} \\ &= \left(\frac{\sqrt{2\kappa}\sigma}{2\pi\sigma^2\sinh\sqrt{2\kappa}\sigma T}\right)^{1/2} \end{aligned} \tag{.6}$$

where $T = t_b - t_a$.

Hence, the kernel can be rewritten as

$$K(b,a) = \left(\frac{\sqrt{2\kappa}\sigma}{2\pi\sigma^2\sinh\sqrt{2\kappa}\sigma(t_b - t_a)}\right)^{1/2} \times \exp\left(-\frac{1}{\sigma^2}\int_{t_a}^{t_b}\left(\frac{1}{2}\dot{\bar{x}}^2(t) + \kappa\sigma^2\bar{x}^2(t)\right)dt\right) \tag{.7}$$

Next, we compute

$$\exp\left(-\frac{1}{\sigma^2}\int_{t_a}^{t_b}\left(\frac{1}{2}\dot{\bar{x}}^2(t) + \kappa\sigma^2\bar{x}^2(t)\right)dt\right)$$

Since the least-action path $\bar{x}(t)$ follows Euler-Lagrange equation, it follows that

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\bar{x}}}\right) - \frac{\partial L}{\partial\bar{x}} = 0$$

associated with the $L = \frac{1}{2}\dot{x}^2(t) + \kappa\sigma^2 x^2(t)$ we get

$$\frac{d\dot{\bar{x}}}{dt} - 2\kappa\sigma^2\bar{x} = 0,$$

or equivalently,

$$\ddot{\bar{x}} = 2\kappa\sigma^2\bar{x}$$

which is an homogeneous linear second-order ODE whose solution can be written as

$$\bar{x}(t) = A\sinh\sigma\sqrt{2\kappa}t + B\cosh\sigma\sqrt{2\kappa}t \tag{.8}$$

Given the solution of $\bar{x}(t)$ and $\dot{\bar{x}}(t)$ above, we can proceed to compute

$$S_{cl} = \int_{t_a}^{t_b}\left(\frac{1}{2}\dot{\bar{x}}^2(t) + \kappa\sigma^2\bar{x}^2(t)\right)dt$$

by simplification first and then direct substitution as follows.

$$\begin{aligned}
S_{cl} &= \int_{t_a}^{t_b} \left(\frac{1}{2}\dot{\bar{x}}^2(t) + \kappa\sigma^2\bar{x}^2(t)\right)dt \\
&= \frac{1}{2}\int_{t_a}^{t_b} \dot{\bar{x}}^2(t)dt + \int_{t_a}^{t_b} \kappa\sigma^2\bar{x}^2(t)dt \\
&= \frac{1}{2}\left([\bar{x}\dot{\bar{x}}]_{t_a}^{t_b} - \int_{t_a}^{t_b} \bar{x}\ddot{\bar{x}}dt\right) + \int_{t_a}^{t_b} \kappa\sigma^2\bar{x}^2(t)dt \\
&= \frac{1}{2}\left([\bar{x}\dot{\bar{x}}]_{t_a}^{t_b} - \int_{t_a}^{t_b} \bar{x}\left(2\kappa\sigma^2\bar{x}\right)dt\right) + \int_{t_a}^{t_b} \kappa\sigma^2\bar{x}^2(t)dt \\
&= \frac{1}{2}[\bar{x}(t)\dot{\bar{x}}(t)]_{t_a}^{t_b}
\end{aligned} \tag{.9}$$

Note that at the two end point that lie on both $\bar{x}(t)$ and $x(t)$ due to the fact $\bar{x}(t)$ and $x(t)$ overlap at the two end point denoted by x_a and x_b, we get

$$x_a = \bar{x}_a = A\sinh\sigma\sqrt{2\kappa}t_a + B\cosh\sigma\sqrt{2\kappa}t_a$$

and

$$x_b = \bar{x}_b = A\sinh\sigma\sqrt{2\kappa}t_b + B\cosh\sigma\sqrt{2\kappa}t_b \tag{.10}$$

from which we can solve for A and B in terms of x_a and x_b, and therefore, finally S_{cl} can be calculated as

$$S_{cl} = \frac{1}{2}\sigma\sqrt{2\kappa}\left[\frac{(x_a^2 + x_b^2)\cosh\sigma\sqrt{2\kappa}(t_b - t_a) - 2x_a x_b}{\sinh\sigma\sqrt{2\kappa}(t_b - t_a)}\right] \tag{.11}$$

The kernel is thus calculated as

$$\begin{aligned}
K(x_b, t_b; x_a, t_a) &= \left[\frac{\sqrt{2\kappa}}{2\pi\sigma\sinh\sqrt{2\kappa}\sigma(t_b - t_a)}\right]^{1/2} \\
&\times \exp\left(-\frac{\sqrt{2\kappa}}{2\sigma}\left[\frac{(x_a^2 + x_b^2)\cosh\sigma\sqrt{2\kappa}(t_b - t_a) - 2x_a x_b}{\sinh\sigma\sqrt{2\kappa}(t_b - t_a)}\right]\right).
\end{aligned} \tag{.12}$$

□

.2 Proof of Proposition 4

The proof takes two steps. Here, we let the quadratic generalized hazard function take the form $\omega(x) = \frac{1}{2}\omega^2 x^2 + \omega_0$ rather than $\Lambda(x) = \kappa x^2$. But in the end a simple change of variables of $\omega = \sqrt{2\kappa}$ and $\omega_0 = 0$ would lead us to the version with $\Lambda(x) = \kappa x^2$.

Step 1: Given the quadratic generalized hazard function $\omega(x) = \frac{1}{2}\omega^2 x^2 + \omega_0$ and the standard deviation of the Brownian motion process Σ for the uncontrolled sticky price gap x, the eigenvalues $-\lambda_n$ is given by

$$-\lambda_n = -\omega\Sigma^2\left(n + \frac{1}{2} + \omega_0\right)$$

and the eigenfunctions $\phi_n(x)$ is given by a Fredholm integral equation of the first kind

$$\int_{-\infty}^{\infty}\left(\frac{\omega}{\pi\Sigma^2}\right)^{1/4} e^{-(\omega/2\Sigma^2)(x-a)^2}\phi_n(x)dx = \left(\frac{\omega}{2\Sigma^2}\right)^{\frac{n}{2}}\frac{a^n}{\sqrt{n!}}e^{-\frac{\omega}{4\Sigma^2}a^2}$$

Proof. To show this, we assume the transition amplitude to go from any state $\psi(x)$ to another state $\chi(x)$ of the sticky price process is denoted by $\langle\chi|1|\psi\rangle$ which is defined by

$$\langle\chi|1|\psi\rangle = \int\int \chi^*(x_b, t_b)K(b,a)\psi(x_a, t_a)dx_a dx_b$$

where $K(b,a)$ denotes the complex version of the transition density from state a to state b.

Suppose $\psi(x)$ and $\chi(x)$ are expanded in terms of the orthogonal functions $\phi_n(x)$, thus we get

$$\psi(x) = \sum_n \psi_n\phi_n(x)$$

$$\chi(x) = \sum_n \chi_n\phi_n(x)$$

It follows from proposition 3 that the its complex counterpart can be written as

$$K(x_b, t_b; x_a, t_a) = \sum_{n=1}^{\infty}\phi_n(x_b)\phi_n^*(x_a)e^{-(i/\Sigma^2)(\lambda_n-\omega_0)(t_b-t_a)}$$

with the generalized hazard function taking the form $\frac{1}{2}\omega^2 x^2 + \omega_0$.

Thus, the transition amplitude can be rewritten as

$$< \chi|1|\psi >= \int\int \chi^*(x_b,t_b)\sum_{n=1}^{\infty}\phi_n(x_b)\phi_n^*(x_a)e^{-(i/\Sigma^2)(\lambda_n-\omega_0)(t_b-t_a)}\psi(x_a,t_a)dx_a dx_b$$

$$= \sum_n \int\int \chi^*(x_b,t_b)\phi_n(x_b)\phi_n^*(x_a)\psi(x_a,t_a)e^{-(i/\Sigma^2)(\lambda_n-\omega_0)(t_b-t_a)}dx_a dx_b$$

$$= \sum_n \Big(\int \chi^*(x_b,t_b)\phi_n(x_b)dx_b\Big)\Big(\int \psi(x_a,t_a)\phi_n^*(x_a)dx_a\Big)e^{-(i/\Sigma^2)(\lambda_n-\omega_0)T}$$

$$= \sum_n \chi_n^*\psi_n e^{-(i/\Sigma^2)(\lambda_n-\omega_0)T}$$

where in the last line we have used $\chi_n^* = \int \chi^*(x)\phi_n(x)dx$ and $\psi_n = \int \psi(x)\phi_n^*(x)dx$ due to the orthogonal functions $\phi_n(x)$ and $T = t_b - t_a$.

Therefore, we get

$$\int\int \chi^*(x_b,t_b)K(b,a)\psi(x_a,t_a)dx_a dx_b = \sum_n \chi_n^*\psi_n e^{-(i/\Sigma^2)(\lambda_n-\omega_0)T}$$

Next, suppose we choose a special pair of functions $\psi(x)$ and $\chi(x)$ for which the expansion on the right hand side of above equation is simple, then after obtaining the functions ψ_n we could get some information about functions $\phi_n(x)$. Suppose we choose the functions $\psi(x)$ and $\chi(x)$ as

$$\psi(x) = \Big(\frac{\omega}{\pi\Sigma^2}\Big)^{1/4}e^{-(\omega/2\Sigma^2)(x-a)^2}$$

$$\chi(x) = \Big(\frac{\omega}{\pi\Sigma^2}\Big)^{1/4}e^{-(\omega/2\Sigma^2)(x-b)^2}$$

These functions above represent Gaussian distributions centered about a and b, respectively. We therefore can set $\psi_n = \psi_n(a)$ and $\chi_n = \psi_n(b)$, then we get

$$\int\int \chi^*(x_b,t_b)K(b,a)\psi(x_a,t_a)dx_a dx_b = \sum_n \psi_n^*(b)\psi_n(a)e^{-(i/\Sigma^2)(\lambda_n-\omega_0)T}$$

We know, from proposition 1, that the complex version of the transition density $K^0(b|a)$ with a quadratic generalized hazard function (in physics they call it harmonic oscillator) is given by

$$K(b,a) = \Big(\frac{\omega}{2\pi i\Sigma^2\sin\omega T}\Big)^{1/2}e^{\frac{i\omega}{2\Sigma^2\sin\omega T}\big((x_b^2+x_a^2)\cos\omega T-2x_b x_a\big)}$$

Plugging in the left hand side of

$$\int\int \chi^*(x_b,t_b)K(b,a)\psi(x_a,t_a)dx_a dx_b = \sum_n \psi_n^*(b)\psi_n(a)e^{-(i/\Sigma^2)(\lambda_n-\omega_0)T},$$

we get

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\frac{\omega}{\pi\Sigma^2}\right)^{1/4}e^{-(\omega/2\Sigma^2)(x_b-b)^2}\left(\frac{\omega}{2\pi i\Sigma^2\sin\omega T}\right)^{1/2}e^{\frac{i\omega}{2\Sigma^2\sin\omega T}\left(\left(x_b^2+x_a^2\right)\cos\omega T-2x_bx_a\right)}$$

$$\times\left(\frac{\omega}{\pi\Sigma^2}\right)^{1/4}e^{-(\omega/2\Sigma^2)(x_a-a)^2}dx_adx_b$$

$$=\left(\frac{\omega}{2\pi i\Sigma^2\sin\omega T}\right)^{1/2}$$

$$\times\left(\frac{\omega}{\pi\Sigma^2}\right)^{1/2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}e^{-(\omega/2\Sigma^2)(x_b-b)^2}e^{\frac{i\omega}{2\Sigma^2\sin\omega T}\left(\left(x_b^2+x_a^2\right)\cos\omega T-2x_bx_a\right)}e^{-(\omega/2\Sigma^2)(x_a-a)^2}dx_adx_b$$

Perform this double Gaussian integral which is somewhat lengthy but direct, we get the result of this integral as

$$\left(\frac{\omega}{2\pi i\Sigma^2\sin\omega T}\right)^{1/2}\left(\frac{\omega}{\pi\Sigma^2}\right)^{1/2}$$

$$\times\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}e^{-(\omega/2\Sigma^2)(x_b-b)^2}e^{\frac{i\omega}{2\Sigma^2\sin\omega T}\left(\left(x_b^2+x_a^2\right)\cos\omega T-2x_bx_a\right)}e^{-(\omega/2\Sigma^2)(x_a-a)^2}dx_adx_b$$

$$=e^{-\frac{i\omega T}{2}-\frac{\omega}{4\Sigma^2}\left(a^2+b^2-2abe^{-i\omega T}\right)}$$

Therefore, we get

$$e^{-\frac{i\omega T}{2}-\frac{\omega}{4\Sigma^2}\left(a^2+b^2-2abe^{-i\omega T}\right)}=\sum_n \psi_n^*(b)\psi_n(a)e^{-(i/\Sigma^2)(\lambda_n-\omega_0)T}$$

Or equivalently,

$$e^{-\frac{\omega}{4\Sigma^2}\left(a^2+b^2\right)}e^{\frac{\omega ab}{2\Sigma^2}e^{-i\omega T}}e^{-\frac{i\omega T}{2}}=\sum_n \psi_n^*(b)\psi_n(a)e^{-(i/\Sigma^2)(\lambda_n-\omega_0)T}$$

Expanding $e^{\frac{\omega ab}{2\Sigma^2}e^{-i\omega T}}$ in powers of $e^{-i\omega T}$ as

$$e^{\frac{\omega ab}{2\Sigma^2}e^{-i\omega T}}=1+\left(\frac{\omega ab}{2\Sigma^2}\right)e^{-i\omega T}+\frac{1}{2!}\left(\frac{\omega ab}{2\Sigma^2}\right)^2e^{-2i\omega T}+\frac{1}{3!}\left(\frac{\omega ab}{2\Sigma^2}\right)^3e^{-3i\omega T}+...$$

Plugging in $e^{-\frac{\omega}{4\Sigma^2}\left(a^2+b^2\right)}e^{\frac{\omega ab}{2\Sigma^2}e^{-i\omega T}}e^{-\frac{i\omega T}{2}} = \sum_n \psi_n^*(b)\psi_n(a)e^{-(i/\Sigma^2)(\lambda_n-\omega_0)T}$, we get

$$e^{-\frac{\omega}{4\Sigma^2}\left(a^2+b^2\right)}\left(1+\left(\frac{\omega ab}{2\Sigma^2}\right)e^{-i\omega T}+\frac{1}{2!}\left(\frac{\omega ab}{2\Sigma^2}\right)^2 e^{-2i\omega T}+...\right)e^{-\frac{i\omega T}{2}}$$

$$= \sum_n \psi_n^*(b)\psi_n(a)e^{-(i/\Sigma^2)(\lambda_n-\omega_0)T}$$

Or equivalently,

$$e^{-\frac{\omega}{4\Sigma^2}\left(a^2+b^2\right)}\left(e^{-\frac{i\omega T}{2}}+\left(\frac{\omega ab}{2\Sigma^2}\right)e^{-\left(1+\frac{1}{2}\right)i\omega T}+\frac{1}{2!}\left(\frac{\omega ab}{2\Sigma^2}\right)^2 e^{-\left(2+\frac{1}{2}\right)i\omega T}+...\right)$$

$$= \sum_n \psi_n^*(b)\psi_n(a)e^{-(i/\Sigma^2)(\lambda_n-\omega_0)T}$$

Comparing terms on both sides, we can solve

$$-\lambda_n = -\omega\Sigma^2\left(n+\frac{1}{2}+\omega_0\right)$$

$$\psi_n(a) = \left(\frac{\omega}{2\Sigma^2}\right)^{\frac{n}{2}}\frac{a^n}{\sqrt{n!}}e^{-\frac{\omega}{4\Sigma^2}a^2}$$

Since $\psi(x) = \sum_n \psi_n \phi_n(x)$, where $\phi_n(x)$ are orthogonal functions, it follows that

$$\int_{-\infty}^{\infty}\left(\frac{\omega}{\pi\Sigma^2}\right)^{1/4}e^{-(\omega/2\Sigma^2)(x-a)^2}\phi_n(x)dx = \left(\frac{\omega}{2\Sigma^2}\right)^{\frac{n}{2}}\frac{a^n}{\sqrt{n!}}e^{-\frac{\omega}{4\Sigma^2}a^2},$$

which is a type of Fredholm integral equation about $\phi_n(x)$ of the first kind that can be solved analytically. □

Step 2: Given eigenvalues

$$-\lambda_n = -\omega\Sigma^2\left(n+\frac{1}{2}+\omega_0\right)$$

and the eigenfunctions $\phi_n(x)$ is given by a Fredholm integral equation of the first kind

$$\int_{-\infty}^{\infty}\left(\frac{\omega}{\pi\Sigma^2}\right)^{1/4}e^{-(\omega/2\Sigma^2)(x-a)^2}\phi_n(x)dx = \left(\frac{\omega}{2\Sigma^2}\right)^{\frac{n}{2}}\frac{a^n}{\sqrt{n!}}e^{-\frac{\omega}{4\Sigma^2}a^2},$$

the eigenvalues $-\lambda_n$ and eigenfunctions $\phi_n(x)$ can be finally written as

$$-\lambda_n = -\omega\Sigma^2\left(n + \frac{1}{2} + \omega_0\right)$$

$$\phi_n(x) = \frac{1}{(2^n n!)^{1/2}}\left(\frac{\omega}{\pi\Sigma^2}\right)^{1/4} H_n\left(\sqrt{\frac{\omega}{\Sigma^2}}x\right)e^{-\left(\frac{\omega}{2\Sigma^2}\right)x^2}$$

where H is the (physicist's) Hermite polynomial of degree n given by $H_n(x) = (-1)^n e^{x^2}\frac{d^n}{dx^n}e^{-x^2}$ and $n = 0, 1, 2, ...$

Proof. Let us rewrite the above integral equation in a form of convolution for the left hand side as

$$\int_{-\infty}^{\infty}\left(\frac{\omega}{\pi\Sigma^2}\right)^{1/4} e^{-(\omega/2\Sigma^2)(a-x)^2}\phi_n(x)dx = \left(\frac{\omega}{2\Sigma^2}\right)^{\frac{n}{2}}\frac{a^n}{\sqrt{n!}}e^{-\frac{\omega}{4\Sigma^2}a^2}$$

Clearly, if let

$$f(a-x) = \left(\frac{\omega}{\pi\Sigma^2}\right)^{1/4} e^{-(\omega/2\Sigma^2)(a-x)^2}$$

$$g(a) = \left(\frac{\omega}{2\Sigma^2}\right)^{\frac{n}{2}}\frac{a^n}{\sqrt{n!}}e^{-\frac{\omega}{4\Sigma^2}a^2},$$

then, the integral equation takes the form of

$$\int_{-\infty}^{\infty} f(a-x)\phi_n(x)dx = g(a),$$

and the left hand side is actually a convolution of $f(x)$ and $\phi_n(x)$. Thus, by taking Fourier transform to both sides of the equation just above, we get

$$\hat{f}(\xi)\hat{\phi}_n(\xi) = \hat{g}(\xi),$$

and therefore,

$$\hat{\phi}_n(\xi) = \frac{\hat{g}(\xi)}{\hat{f}(\xi)},$$

where

$$\hat{f}(\xi) = \int_{-\infty}^{\infty}\left(\frac{\omega}{\pi\Sigma^2}\right)^{1/4} e^{-(\omega/2\Sigma^2)x^2}e^{-i\xi x}dx$$

$$\hat{g}(\xi) = \int_{-\infty}^{\infty}\left(\frac{\omega}{2\Sigma^2}\right)^{\frac{n}{2}}\frac{x^n}{\sqrt{n!}}e^{-\frac{\omega}{4\Sigma^2}x^2}e^{-i\xi x}dx$$

That is,

$$\hat{\phi}_n(\xi) = \frac{\int_{-\infty}^{\infty}\left(\frac{\omega}{2\Sigma^2}\right)^{\frac{n}{2}}\frac{x^n}{\sqrt{n!}}e^{-\frac{\omega}{4\Sigma^2}x^2}e^{-i\xi x}dx}{\int_{-\infty}^{\infty}\left(\frac{\omega}{\pi\Sigma^2}\right)^{1/4}e^{-(\omega/2\Sigma^2)x^2}e^{-i\xi x}dx}$$

With some algebra and simplification, and then taking inverse Fourier transform to $\hat{\phi}_n(\xi)$, we get

$$\phi_n(x) = \frac{1}{\pi\sqrt{2}}\frac{i^n\left(\frac{\omega}{2\Sigma^2}\right)^{\frac{n}{2}}/\sqrt{n!}}{\left(\frac{\omega}{\pi\Sigma^2}\right)^{1/4}}\int_{-\infty}^{\infty}\frac{d^n\left(e^{-\frac{\Sigma^2\xi^2}{\omega}}\right)}{d\xi^n}e^{\frac{\Sigma^2\xi^2}{2\omega}+ix\xi}d\xi$$

Let us first rewrite $\frac{d^n\left(e^{-\frac{\Sigma^2\xi^2}{\omega}}\right)}{d\xi^n}e^{\frac{\Sigma^2\xi^2}{2\omega}}$ in terms of Hermite polynomials $H_n(\sqrt{\frac{\Sigma^2}{\omega}}\xi)$ by the definition of $H_n(\sqrt{\frac{\Sigma^2}{\omega}}\xi) = (-1)^n\frac{d^n\left(e^{-\left(\sqrt{\frac{\Sigma^2}{\omega}}\xi\right)^2}\right)}{d\left(\sqrt{\frac{\Sigma^2}{\omega}}\xi\right)^n}e^{\left(\sqrt{\frac{\Sigma^2}{\omega}}\xi\right)^2}$, which leads to

$$\frac{d^n\left(e^{-\frac{\Sigma^2\xi^2}{\omega}}\right)}{d\xi^n}e^{\frac{\Sigma^2\xi^2}{2\omega}} = \frac{(-1)^n e^{-\left(\sqrt{\frac{\Sigma^2}{2\omega}}\xi\right)^2}H_n(\sqrt{\frac{\Sigma^2}{\omega}}\xi)}{\left(\sqrt{\frac{\omega}{\Sigma^2}}\right)^n}$$

Substituting in the expression for $\phi_n(x)$, we get

$$\phi_n(x) = \frac{1}{\pi\sqrt{2}}\frac{i^n\left(\frac{\omega}{2\Sigma^2}\right)^{\frac{n}{2}}/\sqrt{n!}}{\left(\frac{\omega}{\pi\Sigma^2}\right)^{1/4}}\int_{-\infty}^{\infty}\frac{(-1)^n e^{-\left(\sqrt{\frac{\Sigma^2}{2\omega}}\xi\right)^2}H_n(\sqrt{\frac{\Sigma^2}{\omega}}\xi)}{\left(\sqrt{\frac{\omega}{\Sigma^2}}\right)^n}e^{ix\xi}d\xi$$

Or equivalently,

$$\phi_n(x) = \frac{1}{\pi\sqrt{2}}\frac{i^n\left(\frac{\omega}{2\Sigma^2}\right)^{\frac{n}{2}}/\sqrt{n!}}{\left(\frac{\omega}{\pi\Sigma^2}\right)^{1/4}}\frac{(-1)^n}{\left(\sqrt{\frac{\omega}{\Sigma^2}}\right)^n}\int_{-\infty}^{\infty}e^{-\left(\sqrt{\frac{\Sigma^2}{2\omega}}\xi\right)^2}H_n\left(\sqrt{\frac{\Sigma^2}{\omega}}\xi\right)e^{ix\xi}d\xi$$

On the other hand, by generating function of Hermite Polynomial of $H_n\left(\sqrt{\frac{\Sigma^2}{\omega}}\xi\right)$:

$$e^{2\sqrt{\frac{\Sigma^2}{\omega}}\xi t-t^2} = \sum_{n=0}^{\infty}H_n\left(\sqrt{\frac{\Sigma^2}{\omega}}\xi\right)\frac{t^n}{n!},$$

which can be multiplied by $e^{-\left(\sqrt{\frac{\Sigma^2}{2\omega}}\xi\right)^2}$ on both sides to get

$$e^{-\left(\sqrt{\frac{\Sigma^2}{2\omega}}\xi\right)^2+2\sqrt{\frac{\Sigma^2}{\omega}}\xi t-t^2} = \sum_{n=0}^{\infty} e^{-\left(\sqrt{\frac{\Sigma^2}{2\omega}}\xi\right)^2} H_n\left(\sqrt{\frac{\Sigma^2}{\omega}}\xi\right)\frac{t^n}{n!}$$

Taking inverse Fourier transform of the left hand side, we get

$$\mathcal{F}^{-1}\left(e^{-\left(\sqrt{\frac{\Sigma^2}{2\omega}}\xi\right)^2+2\sqrt{\frac{\Sigma^2}{\omega}}\xi t-t^2}\right)(x)$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-\left(\sqrt{\frac{\Sigma^2}{2\omega}}\xi\right)^2+2\sqrt{\frac{\Sigma^2}{\omega}}\xi t-t^2} e^{ix\xi} d\xi$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-\left(\sqrt{\frac{\Sigma^2}{2\omega}}\xi\right)^2+2\sqrt{\frac{\Sigma^2}{\omega}}\xi t-t^2+ix\xi} d\xi$$

$$= \frac{1}{2\pi}\sqrt{\frac{2\omega\pi}{\Sigma^2}} e^{-\frac{\omega}{2\Sigma^2}x^2} e^{2\sqrt{\frac{\omega}{\Sigma^2}}xit+t^2}$$

$$= \frac{1}{2\pi}\sqrt{\frac{2\omega\pi}{\Sigma^2}} e^{-\frac{\omega}{2\Sigma^2}x^2} \sum_{n=0}^{\infty} H_n\left(\sqrt{\frac{\omega}{\Sigma^2}}x\right)\frac{(it)^n}{n!}$$

That is,

$$\mathcal{F}^{-1}\left(e^{-\left(\sqrt{\frac{\Sigma^2}{2\omega}}\xi\right)^2+2\sqrt{\frac{\Sigma^2}{\omega}}\xi t-t^2}\right)(x) = \frac{1}{2\pi}\sqrt{\frac{2\omega\pi}{\Sigma^2}} e^{-\frac{\omega}{2\Sigma^2}x^2} \sum_{n=0}^{\infty} H_n\left(\sqrt{\frac{\omega}{\Sigma^2}}x\right)\frac{(it)^n}{n!}$$

The inverse Fourier transform of the right hand side of the same equation is

$$\mathcal{F}^{-1}\left(\sum_{n=0}^{\infty} e^{-\left(\sqrt{\frac{\Sigma^2}{2\omega}}\xi\right)^2} H_n\left(\sqrt{\frac{\Sigma^2}{\omega}}\xi\right)\frac{t^n}{n!}\right)(x)$$

$$= \sum_{n=0}^{\infty} \mathcal{F}^{-1}\left(e^{-\left(\sqrt{\frac{\Sigma^2}{2\omega}}\xi\right)^2} H_n\left(\sqrt{\frac{\Sigma^2}{\omega}}\xi\right)\frac{t^n}{n!}\right)(x)$$

$$= \sum_{n=0}^{\infty} \frac{t^n}{n!}\mathcal{F}^{-1}\left(e^{-\left(\sqrt{\frac{\Sigma^2}{2\omega}}\xi\right)^2} H_n\left(\sqrt{\frac{\Sigma^2}{\omega}}\xi\right)\right)(x)$$

$$= \sum_{n=0}^{\infty} \frac{t^n}{n!}\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-\left(\sqrt{\frac{\Sigma^2}{2\omega}}\xi\right)^2} H_n\left(\sqrt{\frac{\Sigma^2}{\omega}}\xi\right) e^{ix\xi} d\xi$$

That is,

$$\mathcal{F}^{-1}\left(\sum_{n=0}^{\infty} e^{-\left(\sqrt{\frac{\Sigma^2}{2\omega}}\xi\right)^2} H_n\left(\sqrt{\frac{\Sigma^2}{\omega}}\xi\right)\frac{t^n}{n!}\right)(x) = \sum_{n=0}^{\infty}\frac{t^n}{n!}\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-\left(\sqrt{\frac{\Sigma^2}{2\omega}}\xi\right)^2} H_n\left(\sqrt{\frac{\Sigma^2}{\omega}}\xi\right)e^{ix\xi}d\xi$$

Since the two inverse Fourier transforms are taken respect to the two sides of the one equation respectively, it follows that

$$\frac{1}{2\pi}\sqrt{\frac{2\omega\pi}{\Sigma^2}}e^{-\frac{\omega}{2\Sigma^2}x^2}\sum_{n=0}^{\infty} H_n\left(\sqrt{\frac{\omega}{\Sigma^2}}x\right)\frac{(it)^n}{n!} = \sum_{n=0}^{\infty}\frac{t^n}{n!}\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-\left(\sqrt{\frac{\Sigma^2}{2\omega}}\xi\right)^2} H_n\left(\sqrt{\frac{\Sigma^2}{\omega}}\xi\right)e^{ix\xi}d\xi,$$

which leads to

$$\frac{t^n}{n!}\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-\left(\sqrt{\frac{\Sigma^2}{2\omega}}\xi\right)^2} H_n\left(\sqrt{\frac{\Sigma^2}{\omega}}\xi\right)e^{ix\xi}d\xi = \frac{1}{2\pi}\sqrt{\frac{2\omega\pi}{\Sigma^2}}e^{-\frac{\omega}{2\Sigma^2}x^2}H_n\left(\sqrt{\frac{\omega}{\Sigma^2}}x\right)\frac{(it)^n}{n!}$$

Or equivalently,

$$\int_{-\infty}^{\infty} e^{-\left(\sqrt{\frac{\Sigma^2}{2\omega}}\xi\right)^2} H_n\left(\sqrt{\frac{\Sigma^2}{\omega}}\xi\right)e^{ix\xi}d\xi = \frac{2\pi n!}{t^n}\frac{1}{2\pi}\sqrt{\frac{2\omega\pi}{\Sigma^2}}e^{-\frac{\omega}{2\Sigma^2}x^2}H_n\left(\sqrt{\frac{\omega}{\Sigma^2}}x\right)\frac{(it)^n}{n!}$$

Note that

$$\phi_n(x) = \frac{1}{\pi\sqrt{2}}\frac{i^n\left(\frac{\omega}{2\Sigma^2}\right)^{\frac{n}{2}}/\sqrt{n!}}{\left(\frac{\omega}{\pi\Sigma^2}\right)^{1/4}}\frac{(-1)^n}{\left(\sqrt{\frac{\omega}{\Sigma^2}}\right)^n}\int_{-\infty}^{\infty} e^{-\left(\sqrt{\frac{\Sigma^2}{2\omega}}\xi\right)^2} H_n\left(\sqrt{\frac{\Sigma^2}{\omega}}\xi\right)e^{ix\xi}d\xi$$

By a direct substitution, we get

$$\phi_n(x) = \frac{1}{\pi\sqrt{2}}\frac{i^n\left(\frac{\omega}{2\Sigma^2}\right)^{\frac{n}{2}}/\sqrt{n!}}{\left(\frac{\omega}{\pi\Sigma^2}\right)^{1/4}}\frac{(-1)^n}{\left(\sqrt{\frac{\omega}{\Sigma^2}}\right)^n}\frac{2\pi n!}{t^n}\frac{1}{2\pi}\sqrt{\frac{2\omega\pi}{\Sigma^2}}e^{-\frac{\omega}{2\Sigma^2}x^2}H_n\left(\sqrt{\frac{\omega}{\Sigma^2}}x\right)\frac{(it)^n}{n!},$$

which can be further simplified as

$$\phi_n(x) = \frac{1}{(2^n n!)^{1/2}}\left(\frac{\omega}{\pi\Sigma^2}\right)^{1/4}H_n\left(\sqrt{\frac{\omega}{\Sigma^2}}x\right)e^{-\left(\frac{\omega}{2\Sigma^2}\right)x^2}$$

To summarize, the eigenvalues $-\lambda_n$ and the corresponding eigenfunctions $\phi_n(x)$ as follows:

$$-\lambda_n = -\omega\Sigma^2\left(n + \frac{1}{2} + \omega_0\right)$$

$$\phi_n(x) = \frac{1}{(2^n n!)^{1/2}} \left(\frac{\omega}{\pi \Sigma^2}\right)^{1/4} H_n\left(\sqrt{\frac{\omega}{\Sigma^2}} x\right) e^{-\left(\frac{\omega}{2\Sigma^2}\right)x^2}$$

which are the desired eigenvalues and eigenfunctions after a simple change of variables.

□