ONLINE APPENDIX FOR:
"General Theory of Sticky Prices and Optimal Monetary Policy with Path Integrals"*

Mingyi Yang†

Washington State University

June 8, 2024

A Appendix

.1 Proof of Proposition 2

Proof. Define $L(\dot{x}, x, t) = \frac{1}{2}\dot{x}^2 + \sigma^2\Lambda(x, t)$. Then,

$$L(\dot{x}, x, t) = \frac{1}{2}\dot{x}^2 + \kappa\sigma^2 x^2 - \sigma^2\left[f(t) - \frac{\mu'(t)}{\sigma^2}\right]x + \frac{\sigma^2 f^2(t)}{4\kappa} + \frac{1}{2}\mu^2(t)$$

and we have, by defining $S[x(t)] = \int_{t_a}^{t_b} L(\dot{x}, x, t)dt$,

$$S[x(t)] = S[\bar{x}(t) + y(t)]$$

*Online Appendix to be posted online.

†School of Economic Sciences, Washington State University in Pullman. Mingyi Yang: mingyi.yang2@wsu.edu.

that is,

$$
\begin{aligned}
I &= S[x(t)] \\
&= \int_{t_a}^{t_b} \left(\frac{1}{2}(\dot{\bar{x}}^2 + 2\dot{\bar{x}}\dot{y} + \dot{y}^2) + \kappa\sigma^2(\bar{x}(t) + y(t))^2 - \sigma^2 \left[f(t) - \frac{\mu'(t)}{\sigma^2} \right] (\bar{x}(t) + y(t)) \right) dt \\
&+ \int_{t_a}^{t_b} \left(\frac{\sigma^2 f^2(t)}{4\kappa} + \frac{\mu^2(t)}{2} \right) dt \\
&= \int_{t_a}^{t_b} \left(\frac{1}{2}\dot{\bar{x}}^2(t) + \kappa\sigma^2\bar{x}^2(t) - \sigma^2 \left[f(t) - \frac{\mu'(t)}{\sigma^2} \right] \bar{x}(t) \right) dt \\
&+ \int_{t_a}^{t_b} \left(\dot{\bar{x}}(t)\dot{y}(t) + \frac{1}{2}\dot{y}^2(t) + 2\kappa\sigma^2\bar{x}(t)y(t) + \kappa\sigma^2 y^2(t) - \sigma^2 \left[f(t) - \frac{\mu'(t)}{\sigma^2} \right] y(t) \right) dt \\
&+ \frac{\sigma^2}{4\kappa} \int_{t_a}^{t_b} f^2(t)dt + \frac{1}{2} \int_{t_a}^{t_b} \mu^2(t)dt
\end{aligned}
$$

Note that

$$
\begin{aligned}
S_1 &= \int_{t_a}^{t_b} \left(\dot{\bar{x}}(t)\dot{y}(t) + 2\kappa\sigma^2\bar{x}(t)y(t) - \sigma^2 \left[f(t) - \frac{\mu'(t)}{\sigma^2} \right] y(t) \right) dt \\
&= \int_{t_a}^{t_b} \dot{\bar{x}}(t)dy(t) + 2\kappa\sigma^2 \int_{t_a}^{t_b} \bar{x}(t)y(t)dt - \int_{t_a}^{t_b} \sigma^2 \left[f(t) - \frac{\mu'(t)}{\sigma^2} \right] y(t)dt \\
&= [\dot{\bar{x}}(t)y(t)]_{t_a}^{t_b} - \int_{t_a}^{t_b} \ddot{\bar{x}}(t)y(t)dt + 2\kappa\sigma^2 \int_{t_a}^{t_b} \bar{x}(t)y(t)dt - \int_{t_a}^{t_b} \sigma^2 \left[f(t) - \frac{\mu'(t)}{\sigma^2} \right] y(t)dt \\
&= -\int_{t_a}^{t_b} \left(2\kappa\sigma^2\bar{x}(t) - \sigma^2 \left[f(t) - \frac{\mu'(t)}{\sigma^2} \right] \right) y(t)dt + 2\kappa\sigma^2 \int_{t_a}^{t_b} \bar{x}(t)y(t)dt \\
&- \int_{t_a}^{t_b} \sigma^2 \left[f(t) - \frac{\mu'(t)}{\sigma^2} \right] y(t)dt \\
&= 0
\end{aligned}
$$

where we have used $y(t_a) = y(t_b) = 0$ and from Euler Lagrange equation for $L(\dot{x}, x, t) = \frac{1}{2}\dot{x}^2 + \kappa\sigma^2 x^2 - \sigma^2 \left[f(t) - \frac{\mu'(t)}{\sigma^2} \right] x + \frac{\sigma^2 f^2(t)}{4\kappa} + \frac{1}{2}\mu^2(t)$ to get $\ddot{\bar{x}}(t) = 2\kappa\sigma^2\bar{x}(t) - \sigma^2 \left[f(t) - \frac{\mu'(t)}{\sigma^2} \right]$.

Therefore, we get

$$
\begin{aligned}
S[x(t)] &= S[\bar{x}(t)+y(t)] \\
&= \int_{t_a}^{t_b}\left(\frac{1}{2}\dot{\bar{x}}^2(t)+\kappa\sigma^2\bar{x}^2(t)-\sigma^2\left[f(t)-\frac{\mu'(t)}{\sigma^2}\right]\bar{x}(t)\right)dt+\int_{t_a}^{t_b}\left(\frac{1}{2}\dot{y}^2(t)+\kappa\sigma^2y^2(t)\right)dt \\
&+\frac{\sigma^2}{4\kappa}\int_{t_a}^{t_b}f^2(t)dt+\frac{1}{2}\int_{t_a}^{t_b}\mu^2(t)dt \\
&= S[\bar{x}(t)]+S[y(t)]+\frac{\sigma^2}{4\kappa}\int_{t_a}^{t_b}f^2(t)dt+\frac{1}{2}\int_{t_a}^{t_b}\mu^2(t)dt
\end{aligned}
$$

where

$$
S[\bar{x}(t)] = \int_{t_a}^{t_b}\left(\frac{1}{2}\dot{\bar{x}}^2(t)+\kappa\sigma^2\bar{x}^2(t)-\sigma^2\left[f(t)-\frac{\mu'(t)}{\sigma^2}\right]\bar{x}(t)\right)dt
$$

$$
S[y(t)] = \int_{t_a}^{t_b}\left(\frac{1}{2}\dot{y}^2(t)+\kappa\sigma^2y^2(t)\right)dt
$$

Therefore, we finally get

$$
\begin{aligned}
K(b,a) &= \int_a^b \exp\left(-\frac{1}{\sigma^2}S[x(t)]\right)\mathcal{D}x(t) \\
&= \int_0^0 \exp\left(-\frac{1}{\sigma^2}S[\bar{x}(t)+y(t)]-\frac{1}{4\kappa}\int_{t_a}^{t_b}f^2(t)dt-\frac{1}{2\sigma^2}\int_{t_a}^{t_b}\mu^2(t)dt\right)\mathcal{D}y(t) \\
&= \int_0^0 \exp\left(-\frac{1}{\sigma^2}S[\bar{x}(t)]-\frac{1}{\sigma^2}S[y(t)]-\frac{1}{4\kappa}\int_{t_a}^{t_b}f^2(t)dt-\frac{1}{2\sigma^2}\int_{t_a}^{t_b}\mu^2(t)dt\right)\mathcal{D}y(t) \\
&= \exp\left(-\frac{1}{4\kappa}\int_{t_a}^{t_b}f^2(t)dt\right)\exp\left(-\frac{1}{2\sigma^2}\int_{t_a}^{t_b}\mu^2(t)dt\right) \\
&\times \exp\left(-\frac{1}{\sigma^2}S[\bar{x}(t)]\right)\int_0^0 \exp\left(-\frac{1}{\sigma^2}S[y(t)]\right)\mathcal{D}y(t)
\end{aligned}
$$

That is, given the generalized hazard function with transitional inflation, the corresponding kernel is given by

$$
\begin{aligned}
K(b,a) = \exp&\left(-\frac{1}{2\sigma^2}\int_{t_a}^{t_b}\mu^2(t)dt\right)\exp\left(\frac{\mu(t)}{\sigma^2}x_b\right)\exp\left(-\frac{1}{4\kappa}\int_{t_a}^{t_b}f^2(t)dt\right) \\
&\times \exp\left(-\frac{1}{\sigma^2}S[\bar{x}(t)]\right)\int_0^0 \exp\left(-\frac{1}{\sigma^2}S[y(t)]\right)\mathcal{D}y(t)
\end{aligned}
$$

where

$$
S[\bar{x}(t)] = \int_{t_a}^{t_b}\left(\frac{1}{2}\dot{\bar{x}}^2(t)+\kappa\sigma^2\bar{x}^2(t)-\sigma^2\left[f(t)-\frac{\mu'(t)}{\sigma^2}\right]\bar{x}(t)\right)dt
$$

$$S[y(t)] = \int_{t_a}^{t_b} \left(\frac{1}{2}\dot{y}^2(t) + \kappa\sigma^2 y^2(t) \right) dt$$

First, we can compute $\int_0^0 \exp\left(-\frac{1}{\sigma^2} S[y(t)]\right) \mathcal{D}y(t)$ using the Fourier series method, and it turns out

$$\begin{aligned}\int_0^0 \exp\left(-\frac{1}{\sigma^2} S[y(t)]\right) \mathcal{D}y(t) &= \int_0^0 \exp\left(-\frac{1}{\sigma^2} \int_{t_a}^{t_b} \left(\frac{1}{2}\dot{y}^2(t) + \kappa\sigma^2 y^2(t) \right) dt\right) \mathcal{D}y(t) \\ &= \left(\frac{\sqrt{2\kappa}\sigma}{2\pi\sigma^2 \sinh \sqrt{2\kappa}\sigma(t_b - t_a)} \right)^{1/2}\end{aligned}$$

To calculate $\int_0^0 \exp\left(-\frac{1}{\sigma^2} \int_{t_a}^{t_b} \left(\frac{1}{2}\dot{y}^2(t) + \kappa\sigma^2 y^2(t)\right) dt\right) \mathcal{D}y(t)$, we first note that the path $y(t)$ has to meet the following requirement: $y(t_a = 0) = y(t_b = T) = 0$, and thus we can write $y(t)$ using Fourier series expansion as

$$y(t) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi t}{T}\right) \tag{.1}$$

Next, by direct plugging in and assuming that the time T is divided into discrete steps of length ϵ, our target of equation can be rewritten as

$$\begin{aligned}
F(T) &= \int_0^0 \exp\left(-\frac{1}{\sigma^2}\int_{t_a}^{t_b}\left(\frac{1}{2}\dot{y}^2(t) + \kappa\sigma^2 y^2(t)\right)dt\right)\mathcal{D}y(t) \\
&= J\frac{1}{A}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\exp\{-\frac{1}{2\sigma^2}\frac{T}{2}\sum_{n=1}^{N}\left[\left(\frac{n\pi}{T}\right)^2 + 2\kappa\sigma^2\right]a_n^2\} \\
&\times \frac{da_1}{A}\frac{da_2}{A}\cdots\frac{da_N}{A} \\
&= \frac{J}{A}\prod_{n=1}^{N}\int_{-\infty}^{\infty}\exp\{-\frac{1}{2\sigma^2}\frac{T}{2}\sum_{n=1}^{N}\left[\left(\frac{n\pi}{T}\right)^2 + 2\kappa\sigma^2\right]a_n^2\}\frac{da_n}{A} \\
&\propto \prod_{n=1}^{N}\left(\frac{n^2\pi^2}{T^2} + 2\kappa\sigma^2\right)^{-1/2} \\
&= \prod_{n=1}^{N}\left(\frac{n^2\pi^2}{T^2}\right)^{-1/2}\prod_{n=1}^{N}\left(1 + \frac{2\kappa\sigma^2 T^2}{n^2\pi^2}\right)^{-1/2} \\
&\propto \left(\frac{\sinh\sqrt{2\kappa}\sigma T}{\sigma\sqrt{2\kappa}T}\right)^{-1/2}
\end{aligned} \tag{.2}$$

where we have applied Euler formula to the derivation from the second-to-last line to the last line.

$F(T)$ can be written in the form

$$F(T) = C\left(\frac{\sinh\sqrt{2\kappa}\sigma T}{\sigma\sqrt{2\kappa}T}\right)^{-1/2} \tag{.3}$$

We consider the case in which $\sqrt{2\kappa}\sigma = 0$, since we already know from the previous derivations about the equivalence of path integral and KFE formulations that $F(T) = \left(\frac{1}{2\pi\sigma^2 T}\right)^{1/2}$ when $\sqrt{2\kappa}\sigma = 0$, which is just the inverse of the normalizing factor A. On the other hand, we also have (by utilizing L'Hopital's rule),

$$\left(\frac{1}{2\pi\sigma^2 T}\right)^{1/2} = \lim_{\sqrt{2\kappa}\sigma\to 0} F(T) = \lim_{\sqrt{2\kappa}\sigma\to 0} C\left(\frac{\sinh\sqrt{2\kappa}\sigma T}{\sigma\sqrt{2\kappa}T}\right)^{-1/2} = C \tag{.4}$$

Therefore, our desired integral $F(T)$ is equal to

$$\begin{aligned} F(T) &= \left(\frac{1}{2\pi\sigma^2 T}\right)^{1/2} \left(\frac{\sinh\sqrt{2\kappa}\sigma T}{\sigma\sqrt{2\kappa}T}\right)^{-1/2} \\ &= \left(\frac{\sqrt{2\kappa}\sigma}{2\pi\sigma^2 \sinh\sqrt{2\kappa}\sigma T}\right)^{1/2} \end{aligned} \tag{.5}$$

where $T = t_b - t_a$.

Hence, the kernel can be rewritten as

$$\begin{aligned} K(b,a) = &\left(\frac{\sqrt{2\kappa}\sigma}{2\pi\sigma^2 \sinh\sqrt{2\kappa}\sigma(t_b - t_a)}\right)^{1/2} \exp\left(-\frac{1}{2\sigma^2}\int_{t_a}^{t_b} \mu^2(t)dt\right) \exp\left(-\frac{1}{4\kappa}\int_{t_a}^{t_b} f^2(t)dt\right) \\ &\times \exp\left(-\frac{1}{\sigma^2}\int_{t_a}^{t_b}\left(\frac{1}{2}\dot{\bar{x}}^2(t) + \kappa\sigma^2\bar{x}^2(t) - \sigma^2\left[f(t) - \frac{\mu'(t)}{\sigma^2}\right]\bar{x}(t)\right)dt\right) \end{aligned}$$

Next, we compute

$$\exp\left(-\frac{1}{\sigma^2}\int_{t_a}^{t_b}\left(\frac{1}{2}\dot{\bar{x}}^2(t) + \kappa\sigma^2\bar{x}^2(t) - \sigma^2\left[f(t) - \frac{\mu'(t)}{\sigma^2}\right]\bar{x}(t)\right)dt\right)$$

Since the least-action path $\bar{x}(t)$ follows Euler-Lagrange equation, it follows that

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\bar{x}}}\right) - \frac{\partial L}{\partial \bar{x}} = 0$$

associated with the $L = \frac{1}{2}\dot{x}^2(t) + \kappa\sigma^2 x^2(t) - \sigma^2\left[f(t) - \frac{\mu'(t)}{\sigma^2}\right]x(t) + \frac{\sigma^2 f^2(t)}{4\kappa} + \frac{1}{2}\mu^2(t)$ we get

$$\frac{d\dot{\bar{x}}}{dt} - 2\kappa\sigma^2\bar{x} + \sigma^2\left[f(t) - \frac{\mu'(t)}{\sigma^2}\right] = 0,$$

or equivalently,

$$\ddot{\bar{x}} = 2\kappa\sigma^2\bar{x} - \sigma^2\left[f(t) - \frac{\mu'(t)}{\sigma^2}\right]$$

which is an inhomogeneous linear second-order ODE whose solution can be written as

$$\begin{aligned} \bar{x}(t) &= A\sinh\{\sigma\sqrt{2\kappa}(t - t_a)\} + B\cosh\{\sigma\sqrt{2\kappa}(t_b - t)\} \\ &- \frac{1}{\sigma\sqrt{2\kappa}}\int_{t_a}^{t}\sigma^2\left[f(s) - \frac{\mu'(s)}{\sigma^2}\right]\sin\sigma\sqrt{2\kappa}(t - s)ds \end{aligned} \tag{.6}$$

Given the solution of $\bar{x}(t)$, we can proceed to compute

$$S_{cl} = \int_{t_a}^{t_b} \left(\frac{1}{2}\dot{\bar{x}}^2(t) + \kappa\sigma^2\bar{x}^2(t) - \sigma^2\left[f(t) - \frac{\mu'(t)}{\sigma^2} \right]\bar{x}(t) \right) dt$$

by simplification first and then direct substitution as follows.

$$\begin{aligned}
S_{cl} &= \int_{t_a}^{t_b} \left(\frac{1}{2}\dot{\bar{x}}^2(t) + \kappa\sigma^2\bar{x}^2(t) - \sigma^2\left[f(t) - \frac{\mu'(t)}{\sigma^2} \right]\bar{x}(t) \right) dt \\
&= \frac{1}{2}\int_{t_a}^{t_b} \dot{\bar{x}}^2(t)dt + \int_{t_a}^{t_b} \kappa\sigma^2\bar{x}^2(t)dt - \int_{t_a}^{t_b} \sigma^2\left[f(t) - \frac{\mu'(t)}{\sigma^2} \right]\bar{x}(t)dt \\
&= \frac{1}{2}\left([\bar{x}\dot{\bar{x}}]_{t_a}^{t_b} - \int_{t_a}^{t_b} \bar{x}\ddot{\bar{x}}dt \right) + \int_{t_a}^{t_b} \kappa\sigma^2\bar{x}^2(t)dt - \int_{t_a}^{t_b} \sigma^2\left[f(t) - \frac{\mu'(t)}{\sigma^2} \right]x(t)dt \\
&= \frac{1}{2}\left([\bar{x}\dot{\bar{x}}]_{t_a}^{t_b} - \int_{t_a}^{t_b} \bar{x}\left(2\kappa\sigma^2\bar{x} - \sigma^2\left[f(t) - \frac{\mu'(t)}{\sigma^2} \right] \right) dt \right) + \int_{t_a}^{t_b} \kappa\sigma^2\bar{x}^2(t)dt \\
&- \int_{t_a}^{t_b} \sigma^2\left[f(t) - \frac{\mu'(t)}{\sigma^2} \right]\bar{x}(t)dt \\
&= \frac{1}{2}[\bar{x}(t)\dot{\bar{x}}(t)]_{t_a}^{t_b} - \frac{1}{2}\int_{t_a}^{t_b} \sigma^2\left[f(t) - \frac{\mu'(t)}{\sigma^2} \right]\bar{x}(t)dt
\end{aligned} \tag{.7}$$

Hence, it follows from the fact $\bar{x}_a = x_a$ and $\bar{x}_b = x_b$ that S_{cl} can be written as

$$\begin{aligned}
S_{cl} &= \frac{1}{2}\sigma\sqrt{2\kappa}\left[\frac{(x_a^2 + x_b^2)\cosh\sigma\sqrt{2\kappa}T - 2x_a x_b}{\sinh\sigma\sqrt{2\kappa}T} \right] \\
&+ \frac{\sigma\sqrt{2\kappa}x_b}{2\sinh\sigma\sqrt{2\kappa}T}\int_{t_a}^{t_b}\left[f(t) - \frac{\mu'(t)}{\sigma^2} \right]\sin\sigma\sqrt{2\kappa}(t - t_a)dt \\
&+ \frac{\sigma\sqrt{2\kappa}x_a}{2\sinh\sigma\sqrt{2\kappa}T}\int_{t_a}^{t_b}\left[f(t) - \frac{\mu'(t)}{\sigma^2} \right]\sin\sigma\sqrt{2\kappa}(t_b - t)dt \\
&- \frac{\sqrt{2\kappa}}{2\sigma\kappa\sinh\sigma\sqrt{2\kappa}T} \\
&\times \int_{t_a}^{t_b}\int_{t_a}^{t}\left[f(t) - \frac{\mu'(t)}{\sigma^2} \right]\left[f(s) - \frac{\mu'(s)}{\sigma^2} \right]\sin\sigma\sqrt{2\kappa}(t_b - t)\sin\sigma\sqrt{2\kappa}(s - t_a)dsdt
\end{aligned} \tag{.8}$$

The kernel is thus calculated as

$$K(b,a)=\left(\frac{\sqrt{2\kappa}\sigma}{2\pi\sigma^2\sinh\sqrt{2\kappa}\sigma(t_b-t_a)}\right)^{1/2}\exp\left(-\frac{2\mu(t)}{\sigma^2}x_b\right)\exp\left(-\frac{1}{2\sigma^2}\int_{t_a}^{t_b}\mu^2(t)dt\right)$$
$$\times\exp\left(-\frac{1}{4\kappa}\int_{t_a}^{t_b}f^2(t)dt\right)\exp\{-\frac{1}{\sigma^2}S_{cl}\},\tag{.9}$$

where $T=t_b-t_a$. □

.2 Proof of Proposition 3

Proof. We first write $K^{\mu(t)}(y|x)$ in terms of $K^0(y|x)$ as

$$\begin{aligned}K^{\mu(t)}(y|x)=&\,e^{-\frac{1}{2\sigma^2}\int_0^t\mu^2(r)dr}e^{-\frac{1}{4\kappa}\int_0^t f^2(r)dr}\\&e^{\frac{\sqrt{2\kappa}}{2\sigma^3\kappa\sinh\sigma\sqrt{2\kappa}t}\int_0^t\int_0^r\left[f(r)+\frac{\mu'(r)}{\sigma^2}\right]\left[f(s)+\frac{\mu'(s)}{\sigma^2}\right]\sin\sigma\sqrt{2\kappa}(t-r)\sin\sigma\sqrt{2\kappa}sdsdr}\\&e^{-\frac{2\mu(t)}{\sigma^2}y+\frac{\sqrt{2\kappa}y}{2\sigma\sinh\sigma\sqrt{2\kappa}t}\int_0^t\left[f(r)+\frac{\mu'(r)}{\sigma^2}\right]\sin\sigma\sqrt{2\kappa}rdr}\\&e^{\frac{\sqrt{2\kappa}x}{2\sigma\sinh\sigma\sqrt{2\kappa}t}\int_0^t\left[f(r)+\frac{\mu'(r)}{\sigma^2}\right]\sin\sigma\sqrt{2\kappa}(t-r)dr}K^0(y|x),\end{aligned}\tag{.10}$$

where we can rewrite $K^0(y|x)$ in terms of the eigenvalue-eigenfunction decomposed form as

$$K^0(y|x)=\sum_{j=1}^{\infty}e^{-\lambda_j t}\phi_j(x)\phi_j(y)\tag{.11}$$

where λ_j and $\phi_j(\cdot)$ are the eigenvalues and corresponding eigenfunctions, respectively. As a result, $K^{\mu(t)}(y|x)$ can be rewritten as

$$\begin{aligned}K^{\mu(t)}(y|x)=&\,e^{-\frac{1}{2\sigma^2}\int_0^t\mu^2(r)dr}e^{-\frac{1}{4\kappa}\int_0^t f^2(r)dr}\\&e^{\frac{\sqrt{2\kappa}}{2\sigma^3\kappa\sinh\sigma\sqrt{2\kappa}t}\int_0^t\int_0^r\left[f(r)+\frac{\mu'(r)}{\sigma^2}\right]\left[f(s)+\frac{\mu'(s)}{\sigma^2}\right]\sin\sigma\sqrt{2\kappa}(t-r)\sin\sigma\sqrt{2\kappa}sdsdr}\\&e^{-\frac{2\mu(t)}{\sigma^2}y+\frac{\sqrt{2\kappa}y}{2\sigma\sinh\sigma\sqrt{2\kappa}t}\int_0^t\left[f(r)+\frac{\mu'(r)}{\sigma^2}\right]\sin\sigma\sqrt{2\kappa}rdr}\\&e^{\frac{\sqrt{2\kappa}x}{2\sigma\sinh\sigma\sqrt{2\kappa}t}\int_0^t\left[f(r)+\frac{\mu'(r)}{\sigma^2}\right]\sin\sigma\sqrt{2\kappa}(t-r)dr}\sum_{j=1}^{\infty}e^{-\lambda_j t}\phi_j(x)\phi_j(y),\end{aligned}\tag{.12}$$

where $K^0(y|x) = \sum_{j=1}^{\infty} e^{-\lambda_j t}\phi_j(x)\phi_j(y)$ solves

$$\partial_t K^0(y|x) = (\sigma^2/2)\partial_y^2 K^0(y|x) - \Lambda(y)K^0(y|x). \tag{.13}$$

To obtain our desired transition density of price gap in the presence of time-varying inflation with firm's reinjection, $\mathcal{K}^{\mu(t)}(y|x)$, we just need to replace

$$\sum_{j=1}^{\infty} e^{-\lambda_j t}\phi_j(x)\phi_j(y)$$

in the expression of $K^{\mu(t)}(y|x)$ by the solution $\mathcal{Q}^0(y|x)$ which solves

$$\partial_t \mathcal{Q}^0(y|x) = (\sigma^2/2)\partial_y^2 \mathcal{Q}^0(y|x) - \Lambda(y)\mathcal{Q}^0(y|x) + \Lambda(y)\delta_{y^*(\tau)}(y). \tag{.14}$$

Given $\mathcal{Q}_0^0(x|x) = \phi_j(x)$ for a same reason as in the case of zero inflation, the solution $\mathcal{Q}^0(y|x)$ takes the form

$$\mathcal{Q}^0(y|x) = \sum_{j=1}^{\infty}\left[e^{-\lambda_j t} + \Lambda(x^*(\tau))\phi_j(x^*(\tau))\int_0^t e^{\lambda_j(\tau-t)}d\tau\right]\phi_j(x)\phi_j(y), \tag{.15}$$

and thus our desired transition density of price gap in the presence of time-varying inflation with firm's reinjection, $\mathcal{K}^{\mu(t)}(y|x)$, is written as

$$\begin{aligned}\mathcal{K}^{\mu(t)}(y|x) = {} & e^{-\frac{1}{2\sigma^2}\int_0^t \mu^2(r)dr} e^{-\frac{1}{4\kappa}\int_0^t f^2(r)dr} \\ & e^{\frac{\sqrt{2\kappa}}{2\sigma^3\kappa\sinh\sigma\sqrt{2\kappa}t}\int_0^t\int_0^r\left[f(r)+\frac{\mu'(r)}{\sigma^2}\right]\left[f(s)+\frac{\mu'(s)}{\sigma^2}\right]\sin\sigma\sqrt{2\kappa}(t-r)\sin\sigma_0\sqrt{2\kappa}s\,ds\,dr} \\ & e^{-\frac{2\mu(t)}{\sigma^2}y+\frac{\sqrt{2\kappa}y}{2\sigma\sinh\sigma\sqrt{2\kappa}t}\int_0^t\left[f(r)+\frac{\mu'(r)}{\sigma^2}\right]\sin\sigma\sqrt{2\kappa}r\,dr} \\ & e^{\frac{\sqrt{2\kappa}x}{2\sigma\sinh\sigma\sqrt{2\kappa}t}\int_0^t\left[f(r)+\frac{\mu'(r)}{\sigma^2}\right]\sin\sigma_0\sqrt{2\kappa}(t-r)dr} \\ & \times\sum_{j=1}^{\infty}\left[e^{-\lambda_j t} + \Lambda(x^*(\tau))\phi_j(x^*(\tau))\int_0^t e^{\lambda_j(\tau-t)}d\tau\right]\phi_j(x)\phi_j(y),\end{aligned} \tag{.16}$$

where λ_j, $\phi_j(x)$, and $\phi_j(y)$ are given by

$$\lambda_j = \sigma\sqrt{2\kappa}\left(j-\frac{1}{2}\right),$$

$$\phi_j(x) = \frac{1}{\pi^{1/4}(2^{j-1}(j-1)!)^{1/2}}\left(\frac{2\kappa}{\sigma^2}\right)^{1/8} H_{j-1}\left(\left(\frac{2\kappa}{\sigma^2}\right)^{1/4}x\right)e^{-\left(\frac{\kappa}{2\sigma^2}\right)^{1/2}x^2},$$

and

$$\phi_j(y) = \frac{1}{\pi^{1/4}(2^{j-1}(j-1)!)^{1/2}} \left(\frac{2\kappa}{\sigma^2}\right)^{1/8} H_{j-1}\left(\left(\frac{2\kappa}{\sigma^2}\right)^{1/4} y\right) e^{-\left(\frac{\kappa}{2\sigma^2}\right)^{1/2} y^2},$$

respectively.

$$\begin{aligned}\mathcal{K}^{\mu(t)}(y|x) - K^{\mu(t)}(y|x) = {} & e^{-\frac{1}{2\sigma^2}\int_0^t \mu^2(r)dr} e^{-\frac{1}{4\kappa}\int_0^t f^2(r)dr} \\ & e^{\frac{\sqrt{2\kappa}}{2\sigma^3\kappa \sinh \sigma\sqrt{2\kappa}t}\int_0^t\int_0^r \left[f(r)+\frac{\mu'(r)}{\sigma^2}\right]\left[f(s)+\frac{\mu'(s)}{\sigma^2}\right] \sin\sigma\sqrt{2\kappa}(t-r)\sin\sigma\sqrt{2\kappa}s ds dr} \\ & e^{-\frac{2\mu(t)}{\sigma^2}y+\frac{\sqrt{2\kappa}y}{2\sigma\sinh\sigma\sqrt{2\kappa}t}\int_0^t\left[f(r)+\frac{\mu'(r)}{\sigma^2}\right]\sin\sigma\sqrt{2\kappa}r dr} \\ & e^{\frac{\sqrt{2\kappa}x}{2\sigma\sinh\sigma\sqrt{2\kappa}t}\int_0^t\left[f(r)+\frac{\mu'(r)}{\sigma^2}\right]\sin\sigma\sqrt{2\kappa}(t-r)dr} \\ & \times \sum_{j=1}^{\infty} \Lambda(x^*(\tau))\phi_j(x^*(\tau)) \int_0^t e^{\lambda_j(\tau-t)} d\tau \phi_j(x)\phi_j(y) \neq 0.\end{aligned} \tag{.17}$$

□

.3 Proof of Proposition 5

Proof. Assuming that the time horizon used in the marginal output impulse response is from $t = 0$ to $t = T$, where T can be infinity or strictly less than infinity. That is, $t \in [0, T]$, where $T \in \mathbb{R}^+ \cup \{0, \infty\}$. It is also assumed that the inflation $\mu(t)$ is zero initially at time $t = 0$, i.e., $\mu(0) = 0$. To summarize, $\mu(0) = \mu(T) = 0$, which also implies $f(0) = f(T) = 0$. Therefore, the functions $\mu(t)$ and $f(t)$ over the time horizon $t \in [0, T]$ can be written, without loss of generality, in terms of Fourier series as a function of orthogonal basis $\{\sin\left(\frac{n\pi t}{T}\right), 1\}$ as

$$\mu(t) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi t}{T}\right), \tag{.18}$$

$$f(t) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi t}{T}\right). \tag{.19}$$

Therefore, $\int_0^T \mu^2(t)dt$ and $\int_0^T f^2(t)dt$ in the expression of $\mathcal{Y}^{\mu(t)}(t)$ can be written in terms of Fourier coefficients as

$$\int_0^T \mu^2(t)dt = \frac{T}{2}\sum_{n=1}^{\infty} a_n^2, \tag{.20}$$

$$\int_0^T f^2(t)dt = \frac{T}{2}\sum_{n=1}^{\infty} b_n^2. \tag{.21}$$

Furthermore, the time derivative of $\mu(t)$, $\dot{\mu}(t)$ (i.e., $\mu'(t)$), can also be written in terms of Fourier series as

$$\dot{\mu}(t) = \frac{\pi}{T}\sum_{n=1}^{\infty} na_n \cos\left(\frac{n\pi t}{T}\right). \tag{.22}$$

Consequently, the integrals involving $\mu(t)$ and $f(t)$ in $\mathcal{Y}^{\mu(t)}(t)$ can be expressed in terms of Fourier series or Fourier coefficients as follows:

$$\begin{aligned}
&\frac{\sqrt{2\kappa}}{2\sigma\sinh\sigma\sqrt{2\kappa}T}\int_0^T \left[f(t) + \frac{\dot{\mu}(t)}{\sigma^2}\right]\sin\sigma\sqrt{2\kappa}t dt \\
&= \frac{1}{2}\frac{\sqrt{2\kappa}}{2\sigma\sinh\sigma\sqrt{2\kappa}T}\sum_{n=1}^{\infty} b_n\left[\frac{T}{n\pi - \sigma\sqrt{2\kappa}T}\sin\left(n\pi - \sigma\sqrt{2\kappa}T\right) - \frac{T}{n\pi + \sigma\sqrt{2\kappa}T}\sin\left(n\pi + \sigma\sqrt{2\kappa}T\right)\right] \\
&+ \frac{\pi}{2\sigma^2 T}\frac{\sqrt{2\kappa}}{2\sigma\sinh\sigma\sqrt{2\kappa}T} \\
&\times \sum_{n=1}^{\infty} na_n\left[\frac{T}{n\pi + \sigma\sqrt{2\kappa}T}\left(1 - \cos\left(n\pi + \sigma\sqrt{2\kappa}T\right)\right) + \frac{T}{n\pi - \sigma\sqrt{2\kappa}T}\left(\cos\left(n\pi - \sigma\sqrt{2\kappa}T\right) - 1\right)\right],
\end{aligned} \tag{.23}$$

which is equal to zero when

$$\begin{gathered} T^* = \frac{n\pi}{\sigma\sqrt{2\kappa}}. \\ n = 1, 2, 3, \ldots \end{gathered} \tag{.24}$$

Moreover,

$$
\begin{aligned}
&\frac{\sqrt{2\kappa}}{2\sigma\sinh\sigma\sqrt{2\kappa}T}\int_0^T\left[f(t)+\frac{\dot{\mu}(t)}{\sigma^2}\right]\sin\sigma\sqrt{2\kappa}(T-t)dt\\
&=\frac{1}{2}\frac{\sqrt{2\kappa}}{2\sigma\sinh\sigma\sqrt{2\kappa}T}\sum_{n=1}^{\infty}b_n\left(\frac{T\sin\sigma\sqrt{2\kappa}T}{n\pi+\sigma\sqrt{2\kappa}T}+\frac{T\sin\sigma\sqrt{2\kappa}T}{n\pi-\sigma\sqrt{2\kappa}T}\right)\\
&+\frac{\pi}{2\sigma^2T}\frac{\sqrt{2\kappa}}{2\sigma\sinh\sigma\sqrt{2\kappa}T}\\
&\times\sum_{n=1}^{\infty}na_n\left[\frac{T}{n\pi-\sigma\sqrt{2\kappa}T}\left(\cos\sigma\sqrt{2\kappa}T-\cos n\pi\right)+\frac{T}{n\pi+\sigma\sqrt{2\kappa}T}\left(\cos n\pi-\cos\sigma\sqrt{2\kappa}T\right)\right],
\end{aligned}
\tag{.25}
$$

which is also equal to zero when

$$
T^{*}=\frac{n\pi}{\sigma\sqrt{2\kappa}},(n=1,2,3,...). \tag{.26}
$$

□

.4 Proof of Proposition 6

Proof. We will conduct our analysis by two steps. First, we formulate the time-dependent perturbation of the implied state- and time-dependent generalized hazard function $\Lambda(x,t)$ and show that the path integral formulated transition density can be equivalently rewritten in terms of the infinite sum of the product of $\lambda_{ji}(T)$ which represents the transition element and the eigenfunctions $\phi_j(y)$ and $\phi_i(x)$ with respect to both j and i, where $\lambda_{ji}(T)$ denotes the transition probability of price gap going from state i at time 0 to state j at time T. That is, we aim to show the path integral formulated transition density of price gap from x at time 0 to y at time T following a monetary shock in the presence of time-varying inflation and the implied state- and time-dependent generalized hazard function, $K^{\mu(t)}(y|x)$, can be written as

$$
K^{\mu(t)}(y|x)=\sum_{j=1}^{\infty}\sum_{i=1}^{\infty}\lambda_{ji}(T)\phi_j(y)\phi_i(x). \tag{.27}
$$

In the second step, we aim to show that $\lambda_{ji}(T)$ for $j=1,2,3,...$, and $i=1,2,3,...$, can be written as

$$\lambda_{ji}(T) = \delta_{ji}e^{-\lambda_i T} + \lambda_{ji}^{(1)}(T) + \lambda_{ji}^{(2)}(T) + \cdots , \tag{.28}$$

where $\delta_{ji} = 1$ whenever $j = i$ and $\delta_{ji} = 0$ whenever $j \neq i$. For each $\lambda_{ji}^{(k)}(T)$, where $k \in \{1, 2, 3, ...\}$, we can calculate it analytically. Consequently, the path integral formulated transition density $K^{\mu(t)}(y|x)$ not only in the case of zero inflation but also in the presence of time-varying inflation can be equivalently written in terms of eigenvalue-eigenfunction decomposition as in equation (83), where $\phi_j(y)$ and $\phi_i(x)$ are the eigenfunctions in the case of zero inflation. Overall, we aim to show that the expression

$$K^{\mu(t)}(y|x) = \sum_{j=1}^{\infty}\sum_{i=1}^{\infty}\Big[\delta_{ji}e^{-\lambda_i T} + \lambda_{ji}^{(1)}(T) + \lambda_{ji}^{(2)}(T) + \cdots\Big]\phi_j(y)\phi_i(x) \tag{.29}$$

is legitimate and each component of it is analytically calculable.

Given the transition density of price gap following a monetary shock from x at time 0 to y at time T in the presence of time-varying inflation and an implied state- and time-dependent generalized hazard function $\Lambda(x, t)$ by path integral formulation written as

$$K^{\mu(t)}(y|x) = \int_x^y e^{-\frac{1}{\sigma^2}\int_0^T\left[\frac{1}{2}\dot{z}^2(\tau)+\sigma^2\Lambda(z,\tau)\right]d\tau}\mathcal{D}z(\tau), \tag{.30}$$

and a Taylor expansion of $e^{-\int_0^T \Lambda(z,\tau)d\tau}$ as

$$e^{-\int_0^T \Lambda(z,\tau)d\tau} = 1 - \int_0^T \Lambda(z,\tau)d\tau + \frac{1}{2!}\Big[-\int_0^T \Lambda(z,\tau)d\tau\Big]^2 + \cdots , \tag{.31}$$

$K^{\mu(t)}(y|x)$ can be rewritten as

$$K^{\mu(t)}(y|x) = K^0(y|x) + K^{(1)}(y|x) + K^{(2)}(y|x) + \cdots , \tag{.32}$$

where

$$K^0(y|x) = \int_x^y e^{-\frac{1}{\sigma^2}\int_0^T \frac{1}{2}\dot{z}^2 d\tau}\mathcal{D}z(\tau) \tag{.33}$$

$$K^{(1)}(y|x) = -\int_x^y e^{-\frac{1}{\sigma^2}\int_0^T \frac{1}{2}\dot{z}^2 d\tau}\int_0^T \Lambda(z(s), s)ds\mathcal{D}z(\tau) \tag{.34}$$

$$K^{(2)}(y|x) = \frac{1}{2}\int_x^y e^{-\frac{1}{\sigma^2}\int_0^T \frac{1}{2}\dot{z}^2 d\tau} \int_0^T \Lambda(z(s),s)ds \int_0^T \Lambda(z(r),r)dr\mathcal{D}z(\tau) \tag{.35}$$

and so forth. □

.5 Proof of Proposition 7

Proof. By path integral formulation in its relation to ordinary integral, we can rewrite $K^{(1)}(y|x)$ as (note that the subscript $\Lambda(x)$ will be suppressed)

$$K^{(1)}(y|x) = -\int_0^T \int_{-\infty}^{\infty} K^0(y|z)\Lambda(z,\tau)K^0(z|x)dzd\tau \tag{.36}$$

and apply similar logic to $K^{(2)}(y|x)$.

Now, by plugging all the terms so far obtained in equation (92) and note that $K^0(y|x) = \sum_{i=1}^{\infty} \phi_i(x)\phi_i(y)e^{-\lambda_i T}$ which is the transition density of price gap with zero inflation, we can rewrite $K^{\mu(t)}(y|x)$ as

$$\begin{aligned} K^{\mu(t)}(y|x) = &\sum_{i=1}^{\infty} \phi_i(x)\phi_i(y)e^{-\lambda_i T} \\ &- \sum_{j=1}^{\infty}\sum_{i=1}^{\infty}\int_0^T \int_{-\infty}^{\infty} \phi_j(y)\phi_j(z)e^{-\lambda_j(T-\tau)}\Lambda(z,\tau)\phi_i(z)\phi_i(x)e^{-\lambda_i\tau}dzd\tau \\ &+ \cdots \end{aligned} \tag{.37}$$

It is thus clear that $K^{\mu(t)}(y|x)$ in equation (97) can be written in the form of spectral decomposition as

$$K^{\mu(t)}(y|x) = \sum_{j=1}^{\infty}\sum_{i=1}^{\infty} \lambda_{ji}(T)\phi_j(y)\phi_i(x)$$

as desired, where

$$\lambda_{ji}(T) = \delta_{ji}e^{-\lambda_i T} + \lambda_{ji}^{(1)}(T) + \lambda_{ji}^{(2)}(T) + \cdots$$

$$\begin{aligned} \lambda_{ji}^{(1)}(T) &= -\int_0^T \int_{-\infty}^{\infty} \phi_j(z)\Lambda(z,\tau)\phi_i(z)e^{-\lambda_j(T-\tau)-\lambda_i\tau}dzd\tau \\ &= -e^{-\lambda_j T}\int_0^T \Lambda_{ji}(\tau)e^{(\lambda_j-\lambda_i)\tau}d\tau \end{aligned} \tag{.38}$$

$$\lambda_{ji}^{(2)}(T) = \int_0^T \left[\int_0^\tau \sum_{k=1}^{\infty} e^{-\lambda_j (T-\tau)} \Lambda_{jk}(\tau) e^{-\lambda_k(\tau - s)} \Lambda_{ki}(s) e^{-\lambda_i s} ds \right] d\tau \tag{.39}$$

and so forth, where $\Lambda_{ji}(\tau)$ is called the matrix element of Λ between states i and j and defined as

$$\Lambda_{ji}(\tau) = \int_{-\infty}^{\infty} \phi_j(z) \Lambda(z, \tau) \phi_i(z) dz. \tag{.40}$$

Hence, we have obtained our desired result of expressing the path integral formulated transition density of price gap $K^{\mu(t)}(y|x)$ with time-varying inflation and an implied state- and time-dependent generalized hazard function in terms of the spectral (eigenvalue-eigenfunction) decomposition. Now, to see how the generalization applies to a specific case, we take the unperturbed zero inflation with an implied time-independent quadratic generalized hazard function $\Lambda(x) = \kappa x^2$ and perturb it, so that we get the first-order approximation of the transition density of price gap following a monetary shock in the presence of time-varying inflation $\mu(t)$ with an implied state- and time-dependent quadratic generalized hazard function $\Lambda(x, t)$, $K^{\mu(t)(1)}(y|x)$, written in terms of spectral (eigenvalue-eigenfunction) decomposition as

$$\begin{aligned} K^{\mu(t)(1)}(y|x) &= \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} \lambda_{ji}^{(1)}(T) \phi_j(y) \phi_i(x) \\ &= \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} \left[-e^{-\lambda_j T} \int_0^T \Lambda_{ji}(\tau) e^{(\lambda_j - \lambda_i)\tau} d\tau \right] \phi_j(y) \phi_i(x) \end{aligned} \tag{.41}$$

where

$$\lambda_i = \sigma \sqrt{2\kappa} \left(i - \frac{1}{2} \right), \tag{.42}$$

and

$$\phi_i(x) = \frac{1}{\pi^{1/4} (2^{i-1}(i-1)!)^{1/2}} \left(\frac{2\kappa}{\sigma^2} \right)^{1/8} H_{i-1} \left(\left(\frac{2\kappa}{\sigma^2} \right)^{1/4} x \right) e^{-\left(\frac{\kappa}{2\sigma^2} \right)^{1/2} x^2}, \tag{.43}$$

where $i = 1, 2, 3, ...$ and $H_{i-1}(\cdot)$ is the Hermite polynomial of degree $i - 1$, and

$$\begin{aligned} \Lambda_{ji} &= \int_{-\infty}^{\infty} \phi_j(z) \Lambda(z) \phi_i(z) dz \\ &= \kappa \int_{-\infty}^{\infty} \phi_j(z) z^2 \phi_i(z) dz. \end{aligned} \tag{.44}$$

Now, we can corresponding figure out the time-dependent perturbation solutions to the version with firm's reinjection by simply replacing $e^{-\lambda_j \tau}$ with $a_j(\tau)$ and $e^{-\lambda_i \tau}$ with $a_i(\tau)$ as

$$\lambda_{ji}(T) = \delta_{ji} a_i(T) + \lambda_{ji}^{(1)}(T) + \lambda_{ji}^{(2)}(T) + \cdots$$

$$\begin{aligned} \lambda_{ji}^{(1)}(T) &= -\int_0^T \int_{-\infty}^{\infty} \phi_j(z)\Lambda(z,\tau)\phi_i(z) a_j(T-\tau) a_i(\tau) dz d\tau \\ &= -\int_0^T \Lambda_{ji}(\tau) a_j(T-\tau) a_i(\tau) d\tau \end{aligned} \tag{.45}$$

$$\lambda_{ji}^{(2)}(T) = \int_0^T \left[\int_0^\tau \sum_{k=1}^{\infty} a_j(T-\tau)\Lambda_{jk}(\tau) a_k(\tau - s)\Lambda_{ki}(s) a_i(s) ds \right] d\tau, \tag{.46}$$

where

$$a_j(t) = e^{-\lambda_j t} + \Lambda^* \phi_j^* \int_0^t e^{\lambda_j(\tau - t)} d\tau, \tag{.47}$$

$$a_i(t) = e^{-\lambda_i t} + \Lambda^* \phi_i^* \int_0^t e^{\lambda_i(\tau - t)} d\tau, \tag{.48}$$

$$a_k(t) = e^{-\lambda_k t} + \Lambda^* \phi_k^* \int_0^t e^{\lambda_k(\tau - t)} d\tau. \tag{.49}$$

□